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APPLICATION OF ALGEBRAIC SYSTEMS IN THE THEORY OF FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The paper provides an overview of the author's results in the study of linear functional differential equations of first order $\dot{x}(t)-(\mathcal{F}x)(t)=b(t)$ generated by five families of linear operators $x\to\mathcal{F}x$. In each of five problems (for each $\mathcal{F}$) by immersion of equations from algebra with traditional (pointwise) multiplication into an algebraic system with special multiplication explicit epresentations for the solutions of the equations (in the Cauchy form) are obtained.

Key Words. Algebraic system; Functional differential equation

AMS(MOS) subject classification. 34K05, 34K06, 26A42, 45A05

Introduction. The paper provides an overview of the author's results in the study of linear functional differential equations of first order

$$\dot{x}(t)-(\mathcal{F}x)(t)=b(t) \tag{1}$$

generated by five families of linear operators $x\to\mathcal{F}x$.

In each of five problems (for each $\mathcal{F}$) by immersion of equations from algebra with traditional (pointwise) multiplication into an algebraic system with special multiplication explicit epresentations for the solutions of the equations (in the Cauchy form) are obtained.

An algebraic system is a triple $\langle\,\mathcal{A},\mathcal{O},\mathcal{R}\,\rangle$, where

- $\mathcal{A}$ is a non-empty set (domain);
- $\mathcal{O}$ is a set of operations;
- $\mathcal{R}$ is a set of relations.

In this survey, the domain $\mathcal{A}$ (depending on $\mathcal{F}$) consists from functions (of one or more variables) or from series of functions.

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The collection $\mathcal{O}$ contains three binary operations: standard addition of elements $+$, standard multiplication by scalars $\cdot$ and special multiplication (which we call skew and denote by $*$ or $\circ$), which differs from traditional (pointwise) multiplication. We admit traditional multiplication too.

The set $\mathcal{R}$ of relations (such that $\operatorname{card}\mathcal{R} = 1$ or $\operatorname{card}\mathcal{R} = 2$ depending on operator $\mathcal{F}$) contains special binary integral relations $xR\,y$ between elements $x, y \in \mathcal{A}$. (We say $xR\,y$, if exists special R-integral $\displaystyle\int xR\,y$, generalizing the Riemann–Stieltjes integral $\displaystyle\int xdy$.) We admit the Riemann–Stieltjes integral too.

1. Let $\mu \in \mathbb{C}$, $|\,\mu\,| \leqslant 1$. In [5] we investigated linear system of differential equations with deviating argument

$$\dot{x}(t) - A\,x(\mu t) = b(\mu t), \tag{2}$$

where A is a constant complex $n\times n$-matrix, $b : \mathbb{C} \to \mathbb{C}^n$ is a single-valued function that is analytic at zero. The family of equations (2) belongs to the family of vector pantograph equations [3]. A special algebraic apparatus is proposed for solving both given equation and for the more general equation (3). More precisely, the concept of μ-product $f * g$ of two single-valued functions f and g that are analytic at zero is introduced, and instead of equation (2) we study the equation

$$\dot{x}(t) - A(\mu t) * x(\mu t) = b(\mu t), \tag{3}$$

in which the matrix A and the vector b consist of elements of the space of single-valued functions $\mathbb{C} \to \mathbb{C}$ analytic at zero (A and b consist of elements of domain $\mathcal{A}$ of algebraic system $\langle\,\mathcal{A}, \mathcal{O}, \mathcal{R}\,\rangle$).

Equation (2) is a special case of equation (3). The existence of the fundamental matrix $X(t)$ of order n for the homogeneous equation (3) are proved. For $\mu \neq 0$ the solution of the problem, consisting of equation (3) and the initial condition $x(0) = w$, can be represented in the form

$$x(t) = X(t) * \Big\{X^{-1}(0)\,w + \mu^{-1}\int_0^{\mu t} X^{-1}(s) * b(s)\,ds\Big\},$$

where $X^{-1}(t)$ is the inverse matrix (it was shown that the matrix X is invertible in the algebra generated by μ-multiplication).

2. Let $t, \mu, a, w \in \mathbb{C}$, $q \in \mathbb{N}$. In [6] we investigated the scalar functional differential equation

$$\dot{x}(t) - a\,x(\mu t^q) = b(t), \tag{4}$$

where b is a formal power series with coefficients from $\mathbb{C}$ (b is an element of the domain $\mathcal{A}$). Solution x is also sought in space $\mathcal{A}$. Equation (4) has a deviation of the argument $F(t) = \mu t^q$, and we call it power-law deviation. To present solutions equation (4), special algebraic constructions are applied: for $p \doteq q-1$ we introduce the concept of associative (μ, p)-product of two series f and g from $\mathcal{A}$, which is denoted by $f * g$. It is proved that the series

$$x(t) \doteq C(t,0)\, w + \int_0^t \frac{\partial}{\partial s}\Big(C(t,\tau) * \int_0^s b(\xi)\, d\xi\Big)\Big|_{s=\tau} d\tau$$

is a solution of the problem consisting of equation (4) and the initial condition $x(0) = w$. By $C(t,\tau)$ we denote the the formal power series of two variables, which is an analogue of the Cauchy function of an ordinary differential equation. In other words, if the series $X(\cdot)$ is a nontrivial solution of the homogeneous equation (4), and $X^{-1}(\cdot)$ is the series inverse in the sense of (μ, p)-multiplication (it exists), then $C(t,\tau) \doteq X(t) * X^{-1}(\tau)$. The series $C(t,\tau)$ satisfies the explicit representation

$$C(t,\tau) = \sum_{n=0}^{\infty} \Big[\sum_{k+m=n} (-1)^m\, q^{\binom{m}{2}}\, \frac{a^n}{G_k(q)\, G_m(q)}\, \mu^{\ell_n(q)}\, t^{d_k(q)}\, \tau^{d_n(q)-d_k(q)} \Big],$$

where $\{d_n(q)\}$, $\{\ell_n(q)\}$, $\{G_n(q)\}$ are some integer sequences (see [6]). The identities are proved:

$$C(s,s) \equiv 1, \qquad C(t,s) * C(s,\tau)) \equiv C(t,\tau), \qquad \frac{\partial}{\partial t}\, C(t,\tau)) \equiv a\, C(\mu t^q, \mu \tau^q).$$

3. Let $\alpha, t \in K \doteq [a,b]$; $x, q_i, f \in \mathrm{C}(K;\mathbb{R})$, $F_i \in \mathrm{C}(K;K)$, $i = 1, \ldots, r$, are continuous functions, and all functions q_i have bounded variation. According to [7], [8], [9], [10], [11] family of equations

$$x(t) - \sum_{i=1}^{r} \lambda_i \int_\alpha^t x(F_i(\cdot))\, dq_i = f(t), \quad \lambda_i \in \mathbb{R}, \tag{5}$$

admits an embedding in the family of Φ-integral equations (see below)

$$x(t) - \int_\alpha^t (dQ * x) = f(t). \tag{6}$$

It is pertinent to note that the family of equations (5) includes into itself the initial problem for the generalized scalar pantograph equation

$$\dot{x}(t) - \sum_{i=1}^{r} a_i(t)\, x\big(F_i(t)\big) = b(t). \tag{7}$$

The specificity of equations (5) – (7) is such that all deviating functions F_i are defined on the same segment K and act from it into yourself. This circumstance makes it possible to refuse from setting of the initial functions and from any additional restrictions on deviating functions.

Φ-integral operators $x(t) \to \int_\alpha^t (dQ * x)$ and $y(t) \to \int_\alpha^t (y * dQ)$ are associated with Φ-multiplication $*$, acting in a special algebra generated by semigroup Φ (which, in turn, is generated by algebraic endomorphisms $\varphi_1, \ldots, \varphi_r$: $(\varphi_i x)(\cdot) = x(F_i(\cdot))$). Non-commutative associative Φ-multiplication of series of functions and Riemann–Stieltjes Φ-integrals with series of functions as arguments of integration are defined in space $\mathrm{C}(K^\ell; \mathbb{R})[\Lambda]$.

In other words, the domain $\mathcal{A} \doteq \mathrm{C}(K^\ell; \mathbb{R})[\Lambda]$ in algebraic system consists of formal power series of functions, the components of which are forms of degree k from non-commuting variables $\lambda_1, \ldots, \lambda_r$ with coefficients from function space $\mathrm{C}(\mathrm{K}^\ell; \mathbb{R})$. (Through Λ we denote the language generated by the alphabet $\{\lambda_1, \ldots, \lambda_r\}$.) In this way, in the algebraic system $\langle\, \mathcal{A}, \mathcal{O}, \mathcal{R}\,\rangle$, we use Φ-multiplication $*$ and two relations: we say $du * v$ or $u * dv$ if exists Φ-integral $\int_\alpha^t (du * v)$ or $\int_\alpha^t (u * dv)$ respectively.

Left and right Riemann–Stieltjes Φ-integrals (if they exist) and skew multiplication are related by the formula of integration by parts:

$$\int_\alpha^\beta (du * v) + \int_\alpha^\beta (u * dv) = (u * v)\Big|_\alpha^\beta .$$

As part of the research, we realized a procedure for constructing of fundamental solution $X(\cdot)$ of equation (6) (that is, solution of equation (6), where $f(t) \equiv 1$ — unit of algebra $\mathrm{C}(K^\ell; \mathbb{R})[\Lambda]$ with Φ-multiplication). With respect to skew multiplication $*$, the function $X(\cdot)$ is invertible and generates product $C(t,\tau) = X(t) * X^{-1}(\tau)$. Under certain conditions on parameters of equation (6) the function $C(t,\tau)$ has all the characteristic properties of the Cauchy function:

$$C(s,s) \equiv 1, \qquad C(t,s) * C(s,\tau) \equiv C(t,\tau),$$

$$C(t,\tau) - \int_\tau^t \big(dQ(s) * C(s,\tau)\big) \equiv 1, \qquad C(t,\tau) - \int_\tau^t \big(C(t,s) * dQ(s)\big) \equiv 1.$$

In terms of the algebraic system, we have obtained representation of the general solution of equation (6):

$$x(t) = C(t,\alpha) * f(\alpha) + \int_\alpha^t \big(C(t,s) * df(s)\big).$$

4. The equation

$$\dot{x}(t) = B\big(t, x(t)\big)\,\dot{Q}(t), \tag{8}$$

given in terms of generalized functions, we call by impulsive (see [22, c. 143]). Symbols x and Q denote n-dimensional and m-dimensional vector functions respectively, and the matrix-valued function $B : \Omega \to \mathbb{C}^{n\times m}$ is defined in the domain $\Omega \subseteq \mathbb{R} \times \mathbb{C}^n$. It is assumed that the left and right sides of equation define linear continuous functionals (generalized functions) in the space of basic functions D, and equation (8) itself is understood as mathematical notation of the problem of finding such regulated functions $x(\cdot)$, for which the equality $(\dot{x}, \varphi) = \big(B(\cdot, x)\dot{Q},\, \varphi\big)$ takes place for all $\varphi \in D$.

Let $K \doteq [a, b]$ is the segment, and through $\mathrm{G} \doteq \mathrm{G}[a, b]$ we denote the space of *regulated* (see [4, p. 16]) functions, that is, functions $x : K \to \mathbb{C}$, having finite limits $x(t-0)$ for all $t \in (a, b]$ and $x(t+0)$ for all $t \in [a, b)$. Space G, endowed with the natural operation of multiplication of functions, is Banach algebra in the sup-norm.

Regulated functions have the property that for all points $t \in K$ (with the exception of the extreme ones) three values $x(t-0)$, $x(t)$ and $x(t+0)$ are defined, which allows us to construct other accompanying attributes of functions and to get meaningful new results. In this review (see [12], [13], [14], [15], [16], [17], [18]), we define the concepts adjoint multiplication and adjoint integral, generating algebraic systems $\langle\, \mathrm{G}^T, \mathcal{O}, \mathcal{R}\,\rangle$, $\langle\, \Gamma, \mathcal{O}, \mathcal{R}\,\rangle$ and $\langle\, \mathrm{BV}, \mathcal{O}, \mathcal{R}\,\rangle$ (see below for the definitions of all sets).

Finite or countable set $T \doteq \{\tau_1, \tau_2, \ldots\}$ in pairs different points $\tau_k \in K$ will be called the partition of the segment $K \doteq [a, b]$, and the set of all partitions of the segment K is denoted by $\mathbb{T}(K)$. We also include the empty set in the collection $\mathbb{T}(K)$, — it is the smallest element of a partial order defined on the set $\mathbb{T}(K)$ in a natural way: $T \prec S$, if $T \subseteq S$.

In the algebra G, the parametric lattice $\{\, \mathrm{G}^T \,\}_{T\in\mathbb{T}(K)}$ of subalgebras of a special form and a subalgebra Γ representing their intersection are investigated. Algebra Γ contains the algebra BV of functions of bounded variation. In G^T the projectors $P_{_T} : x \to x_{_T}$ and $P^{^T} : x \to x^{^T}$ are defined. In Γ [and in BV] the projectors $P_c : x \to x_c$ and $P^c : x \to x^c$ are defined. The questions of existence of Riemann–Stieltjes integrals of functions-projections of functions of algebras G^T, Γ and BV are investigated. The completeness of the algebras is proved (each algebra uses own norm).

In the algebra G^T the concepts of adjoint multiplication and adjoint integral are introduced. If $x, y \in \mathrm{G}^T$, then

$$x \cdot y \doteq x^{^T} y^{^T} - x_{_T} y_{_T} \quad \text{and} \quad \int_\alpha^t x \cdot dy \doteq \int_\alpha^t x^{^T} dy^{^T} - \int_\alpha^t x_{_T} dy_{_T}.$$

In the algebra Γ [and in BV] the concepts of adjoint multiplication and adjoint integral are introduced. If $x, y \in \Gamma$ [or $x, y \in \mathrm{BV}$], then

$$x \circ y \doteq x^c y^c - x_c y_c \quad \text{and} \quad \int_\alpha^t x \circ dy \doteq \int_\alpha^t x^c dy^c - \int_\alpha^t x_c dy_c.$$

The adjoint integrals generate binary integral relations $x \cdot dy$ and $x \circ dy$ between elements of algebras (that is, the sets $\mathcal{R}$ in algebraic systems $\langle\, \mathrm{G}^T, \mathcal{O}, \mathcal{R}\,\rangle$, $\langle\, \Gamma, \mathcal{O}, \mathcal{R}\,\rangle$, $\langle\, \mathrm{BV}, \mathcal{O}, \mathcal{R}\,\rangle$).

Further, by $\mathrm{G} \doteq \mathrm{G}(a,b)$ we denote the algebra *of regulated* functions, defined on the interval $K \doteq (a,b)$. For any $x \in \mathrm{G}$ are defined generalized regulated function $\varphi \to (x, \varphi)$ and generalized derivative of regulated functions $\varphi \to (x', \varphi)$. The adjoint integrals generate the adjoint generalized derivatives of regulated functions, respectively $\varphi \to (\dot{x}, \varphi)^T$ and $\varphi \to (\mathring{x}, \varphi)$. Therefore, three types of differential equations of the form (8) given in terms of generalized regulated functions are defined.

The potential of the proposed constructions is demonstrated by the theorem below. In the statement the following notations are used: $T(x)$ is an at most countable set consisting from all points of discontinuity of the function $x \in \mathrm{G}$; for any $M \subseteq K$ the algebra $\mathrm{H}^{\mathrm{loc}}[M]$ consists of jump functions $x : K \to \mathbb{C}$ such that $T(x) \subseteq M$.

Theorem. *Let $\alpha \in K$, $Q \in \mathrm{BV}^{\mathrm{loc}}$, A is complex $n \times n$-matrix and*

$$X = \big\{ x \in \Gamma^{\mathrm{loc}} \colon\ T(x) \cap T(Q) = \varnothing \big\}.$$

For operator $\mathrm{V} :\ X^n \to \Gamma_n^{\mathrm{loc}}$ such that $(\mathrm{V}x)(t) \doteq x(t) - \int_\alpha^t Ax\, dQ$, and for any $y \in \Gamma_n^{\mathrm{loc}}$ the family of solutions of the equation $(\mathring{\mathrm{V}}x, \varphi) \equiv (\mathring{y}, \varphi)$ can be represented as

$$x(t) = e^{AQ^c(t)} \Big[\, h(t) + \int_\alpha^t e^{-AQ^c(\cdot)}\, dy^c \Big] \qquad \forall\, h \in \mathrm{H}_n^{\mathrm{loc}}[\, K \backslash T(Q)].$$

The collection $x(t) = e^{AQ^c(t)} \Big[\, c + \int_\alpha^t e^{-AQ^c(\cdot)}\, dy^c \Big]$, $c \in \mathbb{C}^n$, is the family of all continuous solutions of the equation.

In other words, the representations for $x(\cdot)$ appearing in the statement of the theorem can be considered as solutions of a linear impulsive equation $\dot{x}(t) - A\,\dot{Q}(t)\,x(t) = \dot{y}(t)$ (with constant coefficients) given in terms of generalized regulated functions.

The family of all inextensible solutions of the equation $\dot{x} = x$ has the form $x(t) = c\,e^t,\ t \in K$. *The theorem expands our possibilities*:

$$(\,\mathring{x}, \varphi) \equiv (x, \varphi) \qquad \Longleftrightarrow \qquad x(t) = h(t)\, e^t,\ t \in K, \quad \forall\, h \in \mathrm{H}^{\mathrm{loc}}[\, K].$$

5. For two regulated functions x, y, given on the segment $[a, b]$, and for a special parameter Δ, called by the defect, the concept of a quasi-integral $\int_a^b x\Delta y$ is defined. In other words, we have binary relation $x\Delta y$; in works [19], [20], [21] it generates the algebraic system $\langle\, \mathrm{G}[a,b], \mathcal{O}, \mathcal{R}\,\rangle$.

If there is a Riemann–Stieltjes integral, then for any defect there is a quasi-integral, and they are all equal to each other. The Perron–Stieltjes integral, if it exists, coincides with one of quasi-integrals, where the defect is defined in a special way ($\Delta = \Delta_0$). Necessary and sufficient conditions for the existence of quasi-integrals are obtained, and their basic properties are proved (in particular, an analogue of the formula for integration by parts).

An existence and uniqueness theorem for the solution of the quasi-integral equation

$$x(t) - \lambda A \int_\alpha^t x\Delta Q = y(t), \quad t \in [a, b], \tag{9}$$

is proved (with a constant real $n\times n$-matrix A). The kernel Q of the system is scalar piecewise continuous function of bounded variation, components of vectors x and y are regulated functions, spectral parameter $\lambda \in \mathbb{R}$ is a regular number. Under certain conditions, the quasi-integral equation (9) can be interpreted as the impulsive problem

$$\dot{x}(t) - \lambda A\, \dot{Q}(t)\, x(t) = \dot{y}(t), \quad x(\alpha) = y(\alpha).$$

An explicit representation for the solution of a homogeneous quasi-integral equation is obtained. For an absolutely regular spectral parameter, an analogue of the Cauchy matrix is defined, its properties are investigated and under certain conditions an explicit representation for the solution quasi-integral equation in the Cauchy form is obtained:

$$x(t) = C(t,\alpha)\, y(\alpha) + \int_\alpha^t C(t,s)\Delta^* y(s).$$

(The relation $x\Delta^* y$ is called dual to the relation $x\Delta y$.) Similar results were obtained for the conjugate and adjoint equations. The identities are proved:

$$C(s,s) \equiv E, \qquad C(t,s)\, C(s,\tau) \equiv C(t,\tau),$$

$$C(t,\tau) - \lambda A \int_\tau^t C(s,\tau)\, \Delta Q(s) \equiv E, \qquad C(t,\tau) - \lambda A \int_\tau^t C(t,s)\, \Delta^* Q(s) \equiv E.$$

Most of the author's publications presented in the bibliography below are available (in Russian) on the All-Russian portal Math-Net.Ru at

http://www.mathnet.ru/php/person.phtml?option_lang=rus&personid=42389

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