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ORDINARY DIFFERENTIAL EQUATIONS

Analog of the Cauchy Function for a Generalized Equation with Deviating Argument

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1. INTRODUCTION

The properties of the Cauchy matrix for equations with deviating argument were studied in [1–5] etc. For some special deviations $F(\cdot)$, the solution of the equation

$$x(t) - \lambda \int_{\alpha}^{t} x(F(\cdot))dq = f(t) \tag{1}$$

admits a closed-form representation using the operations of a special function algebra. (The case $F(\xi) = \mu\xi$ was considered in [6], and the case $F(\xi) = \mu\xi^{\nu}$ was studied in [7].) Here $\alpha, t \in D \doteq [a, b]$, $\lambda \in \mathbb{R}$, $x, q, f \in \mathrm{C}(D, \mathbb{R})$ are continuous functions, q is of bounded variation, and $F : D \to D$ is a continuous function. An analysis of this representation necessitates a more general statement of the original equation (1) in the generalized (F-integral) form

$$x(t) - \int_{\alpha}^{t} (dQ * x) = f(t), \tag{2}$$

where the operator $x(t) \to \int_{\alpha}^{t}(dQ * x)$ is closely related to the multiplication $*$ in the algebra generated by the deviation F. The aim of the present paper is to construct the Cauchy series for Eq. (2). The results were partially announced in [8, 9]. We represent the exposition in the scalar setting.

2. F-MULTIPLICATION OF SERIES

We fix a closed interval $D \doteq [a, b]$ and $\ell \in \mathbb{N}$. By $\mathrm{C} \doteq \mathrm{C}\left(D^{\ell}\right)$ we denote the algebra (over the field $\mathbb{R}$) of continuous functions $x : D^{\ell} \to \mathbb{R}$. Further, let $F : D \to D$ be a given continuous function. By $\mathrm{C}[\lambda] \doteq \mathrm{C}\left(D^{\ell}\right)[\lambda]$ we denote the linear space of formal power series (in powers of $\lambda \in \mathbb{R}$) of the form $\sum_k \lambda^k x_k \doteq \sum_{k=0}^{\infty} \lambda^k x_k$ with function coefficients $x_k \in \mathrm{C}$. For arbitrary $x \in \mathrm{C}$ and k, we use the notation

$$x^{[k]} \doteq x^{[k]}(t_1, \ldots, t_\ell) \doteq x\left(F^{[k]}(t_1), \ldots, F^{[k]}(t_\ell)\right),$$

where $F^{[k]}(\xi) \doteq F(F(\ldots F(\xi)\ldots))$ is the k-fold composition of F. It is natural to set $F^{[0]}(\xi) = \xi$; therefore, $x^{[0]} = x$. Obviously, $\left(x^{[k]}\right)^{[m]} = x^{[k+m]}$ for any $k, m = 0, 1, \ldots$

The F-*product* of series $\sum_k \lambda^k x_k$ and $\sum_m \lambda^m y_m$ in the space $\mathrm{C}[\lambda]$ is defined as the series in $\mathrm{C}[\lambda]$ given by the right-hand side of the formula $\sum_k \lambda^k x_k * \sum_m \lambda^m y_m = \sum_n \lambda^n \sum_{k+m=n} x_k y_m^{[k]}$. The binary operation $*$ is referred to as F-*multiplication*.

Theorem 1. *The space* $\mathrm{C}[\lambda]$ *equipped with the operation of* F*-multiplication is a unital associative algebra over* $\mathbb{R}$ *(which will be denoted by* $\mathrm{C}_F[\lambda] \doteq \mathrm{C}_F\left(D^\ell\right)[\lambda]$*).*

Proof. For the series $x \doteq \sum_k \lambda^k x_k$, $y \doteq \sum_m \lambda^m y_m$, and $z \doteq \sum_n \lambda^n z_n$, the expressions $x * y$ and $y * z$ are equal to $\sum_i \lambda^i \sum_{k+m=i} x_k y_m^{[k]}$ and $\sum_i \lambda^i \sum_{m+n=i} y_m z_n^{[m]}$, respectively. Consequently,

$$\begin{aligned}(x * y) * z &= \sum_j \lambda^j \sum_{i+n=j} \sum_{k+m=i} x_k y_m^{[k]} z_n^{[i]} = \sum_j \lambda^j \sum_{k+m+n=j} x_k y_m^{[k]} z_n^{[k+m]} \\ &= \sum_j \lambda^j \sum_{k+i=j} x_k \sum_{m+n=i} y_m^{[k]} z_n^{[k+m]} = \sum_j \lambda^j \sum_{k+i=j} x_k \left(\sum_{m+n=i} y_m z_n^{[m]} \right)^{[k]} \\ &= x * (y * z),\end{aligned}$$

which implies that F-multiplication is associative. The unity is given by the series $e \doteq \sum_n \lambda^n \delta_{n0}$ (where δ_{n0} is the Kronecker delta).

Lemma 1. *The series* $x \doteq \sum_k \lambda^k x_k$ *is invertible in the algebra* $\mathrm{C}_F[\lambda]$ *if and only if the coefficient* x_0 *is invertible in the algebra* C *[which is equivalent to the condition* $x_0(t) \neq 0$ *for all* $t \in D$*].*

The relation $x * y = e$, where $y \doteq \sum_m \lambda^m y_m$, is valid if and only if $x_0 y_n + \sum_{k=1}^n x_k y_{n-k}^{[k]} = \delta_{n0}$ for all $n = 0, 1, \ldots$; therefore, the right invertibility of the series x is equivalent to the invertibility of the element x_0 in C. The left inverse series $z \doteq \sum_n \lambda^n z_n$ is found from the system

$$z_n x_0^{[n]} + \sum_{m=1}^n z_{n-m} x_m^{[n-m]} = \delta_{n0}, \qquad n = 0, 1, \ldots;$$

moreover, since $\mathrm{C}_F[\lambda]$ is an associative algebra, it follows that the series y and z coincide.

Lemma 2. *The series* $1 - \sum_{k=1}^\infty \lambda^k x_k$ *and* $1 + \sum_{m=1}^\infty \lambda^m \sum_{p_1+\cdots+p_r=m} \prod_{i=1}^r x_{p_i}^{[m-p_1-\cdots-p_i]}$ *are mutually inverse in the algebra* $\mathrm{C}_F[\lambda]$*. The inner summation in the second series is over all ordered sets* $(p_1, \ldots, p_r)$ *of nonnegative integers such that* $p_1 + \cdots + p_r = m$*.*

Proof. Let $u_0 = v_0 = 1$, $u_n = -x_n$, $v_n = \sum_{p_1+\cdots+p_r=n} \prod_{i=1}^r x_{p_i}^{[n-p_1-\cdots-p_i]}$, $n \in \mathbb{N}$, $u \doteq \sum_k \lambda^k u_k$, $v \doteq \sum_m \lambda^m v_m$, and $w_n = \sum_{k+m=n} u_k v_m^{[k]}$, $n = 0, 1, \ldots$ Obviously, $w_0 = 1$, $w_1 = u_0 v_1^{[0]} + u_1 v_0^{[1]} = v_1 + u_1 = x_1^{[0]} - x_1 = 0$, and the chain of relations

$$w_n = u_0 v_n^{[0]} + \sum_{p=1}^{n-1} u_p v_{n-p}^{[p]} + u_n v_0^{[n]} = v_n - x_n - \sum_{p=1}^{n-1} x_p \sum_{p_1+\cdots+p_r=n-p} \prod_{i=1}^r x_{p_i}^{[n-p_1-\cdots-p_i]}$$

is valid for $n > 1$. By setting $p_{r+1} = p$ and by passing from repeated summation to simultaneous summation with respect to all variables, we continue the chain of relations:

$$\begin{aligned}w_n &= v_n - x_n - \sum_{\substack{p_1+\cdots+p_{r+1}=n \\ p_{r+1}<n}} \left(\prod_{i=1}^r x_{p_i}^{[n-p_1-\cdots-p_i]} \right) x_{p_{r+1}}^{[n-p_1-\cdots-p_{r+1}]} \\ &= v_n - \sum_{p_1+\cdots+p_j=n} \prod_{i=1}^j x_{p_i}^{[n-p_1-\cdots-p_i]} = 0.\end{aligned}$$

(We have replaced $r+1$ by j and transferred x_n into the sum.) Therefore, $w_n = \delta_{n0}$, whence we obtain $u * v = e$.

Remark 1. The power series $1-\sum_{k=1}^{\infty}\lambda^k a_k$ and $1+\sum_{m=1}^{\infty}\lambda^m\sum_{p_1+\cdots+p_r=m}\prod_{i=1}^{r}a_{p_i}$ with numerical coefficients are mutually invertible in the sense of natural multiplication of series.

3. THE RIEMANN–STIELTJES F-INTEGRAL

Definition 1. We fix $i\in\{1,\dots,\ell\}$, a closed interval $E\subseteq D$, and series $u,v\in \mathrm{C}_F\left(D^\ell\right)[\lambda]$. If, for all $k,m=0,1,\dots$, there exist Riemann–Stieltjes integrals $\int_E\left(u_k\cdot d_i v_m^{[k]}\right)$, then the series

$$\int\limits_E (u*d_i v)\doteq\sum_n\lambda^n\sum_{k+m=n}\int\limits_E\left(u_k\cdot d_i v_m^{[k]}\right)\tag{3}$$

is called the *left F-integral of the series u with respect to the series v over the variable $t_i\in E$*. If, for all $k,m=0,1,\dots$, there exist Riemann–Stieltjes integrals $\int_E\left(d_i u_k\cdot v_m^{[k]}\right)$, then the series

$$\int\limits_E (d_i u*v)\doteq\sum_n\lambda^n\sum_{k+m=n}\int\limits_E\left(d_i u_k\cdot v_m^{[k]}\right)\tag{4}$$

is called the *right F-integral of the series v with respect to the series u over the variable $t_i\in E$.*

Each of the F-integrals (3) and (4) is linear with respect to each of the arguments and satisfies the additivity property (provided that all related F-integrals exist).

Theorem 2. *The existence of one of the F-integrals $\int_\alpha^\beta(u*d_i v)$ and $\int_\alpha^\beta(d_i u*v)$ implies the existence of the other*, *and*

$$\int\limits_\alpha^\beta(u*d_i v)+\int\limits_\alpha^\beta(d_i u*v)=(u*v)\big|_\alpha^\beta,\tag{5}$$

where α and β are substituted for t_i.

The simultaneous existence of the F-integrals (3) and (4) follows from the simultaneous existence of the integrals $\int_\alpha^\beta\left(u_k\cdot d_i v_m^{[k]}\right)$ and $\int_\alpha^\beta\left(d_i u_k\cdot v_m^{[k]}\right)$, and relation (5) follows from the integration by parts formula.

We restrict the investigation of the existence of F-integrals to the following assertion.

Theorem 3. *If*, *for series $u,v\in \mathrm{C}_F\left(D^\ell\right)[\lambda]$, the coefficients u_k of the series u are of bounded variation with respect to the variable t_i, then the F-integrals* (3) *and* (4) *exist for any closed interval $E\subseteq D$.*

Proof. To prove the theorem, it suffices to note that the coefficients of the F-integral (4), that is, the sums $\sum_{k+m=n}\int_E\left(d_i u_k\cdot v_m^{[k]}\right)$, exist, since all functions u_k are of bounded variation with respect to the variable t_i and all function $v_m^{[k]}$ are continuous [10, p. 216].

Lemma 3. *If $u,v,w\in \mathrm{C}_F(D)[\lambda]$ are series such that the coefficients u_k of the series u are of bounded variation, then $\int_E(du(s)*(v(s)*w(\tau)))=\int_E(du*v)*w(\tau)$ for any $E\subseteq D$.*

First, note that the F-multiplication and F-integration on the left-hand side in the formula are performed in the algebra $\mathrm{C}_F\left(D^2\right)[\lambda]$, and those on the right-hand side, in the algebra $\mathrm{C}_F(D)[\lambda]$. The series $v(s)*w(\tau)$ and $\int_E(du*v)$ are equal to

$$\sum_i\lambda^i\sum_{m+n=i}v_m(s)w_n^{[m]}(\tau)\quad\text{and}\quad\sum_i\lambda^i\sum_{k+m=i}\int\limits_E\left(du_k\cdot v_m^{[k]}\right),$$

respectively. Consequently, we have the chain of relations

$$\int\limits_E (du(s) * (v(s) * w(\tau))) = \sum_j \lambda^j \sum_{k+i=j} \int\limits_E \left(du_k(s) \cdot \sum_{m+n=i} v_m^{[k]}(s) \left(w_n^{[m]}\right)^{[k]}(\tau) \right)$$

$$= \sum_j \lambda^j \sum_{k+m+n=j} \int\limits_E \left(du_k \cdot v_m^{[k]}\right) w_n^{[k+m]}(\tau) = \sum_j \lambda^j \sum_{i+n=j} \sum_{k+m=i} \int\limits_E \left(du_k \cdot v_m^{[k]}\right) w_n^{[i]}(\tau)$$

$$= \int\limits_E (du * v) * w(\tau).$$

Remark 2. The above-proved formula implies that a series independent of the integration variable and written on the right can be transferred outside the F-integral. At the same time, one can readily show that, in general, the series $\int_E((u(t) * v(s)) * dw(s))$ and $u(t) * \int_E(v * dw)$ are different.

Example 1. Let $D = [-1, 1]$, and let $F : D \to D$ satisfy the condition $F(\xi) = \mu\xi$, $|\mu| \le 1$. The series $Q(t) = \sum_k \lambda^k \delta_{k1} t$ and $x(t) = \sum_m \lambda^m \mu^{C_m^2} t^m/m!$ satisfy the relation $x(t) - \int_0^t (dQ * x) = e$. Indeed, we have

$$\int\limits_0^t (dQ * x) = \sum_n \lambda^n \sum_{k+m=n} \int\limits_0^t \left(dQ_k \cdot x_m \left(F^{[k]}(\cdot)\right)\right) = \sum_{n=1}^{\infty} \lambda^n \int\limits_0^t (ds \cdot x_{n-1}(\mu s))$$

$$= \sum_{n=1}^{\infty} \lambda^n \int\limits_0^t \frac{1}{(n-1)!} \mu^{C_{n-1}^2} (\mu s)^{n-1} ds = \sum_{n=1}^{\infty} \lambda^n \frac{1}{n!} \mu^{C_n^2} t^n = x(t) - e.$$

The series $Q(t)$ and $x(t)$ are uniformly convergent on D for any $\lambda \in \mathbb{R}$; consequently, the equation $x(t) - \int_0^t (dQ * x) = e$ is a generalization of the equation $\tilde{x}(t) - \lambda \int_0^t \tilde{x}(\mu s) ds = 1$ (see the third element of the chain), or the problem $d\tilde{x}(t)/dt = \lambda \tilde{x}(\mu t)$, $\tilde{x}(0) = 1$. Here $\tilde{x} = \sum_n \lambda^n x_n$ is the sum of the series.

Example 2. We fix the function $F(\xi) = \xi^q$, $q > 0$, defined on the interval $D = [0, 1]$. If $Q(t) = \sum_k \lambda^k \delta_{k1} t^p$ and $x(t) = \sum_m \lambda^m t^{p h_m(q)}/G_m(q)$, where $h_0(q) \doteq 0$, $h_m(q) \doteq \sum_{i=0}^{m-1} q^i$, $G_0(q) \doteq 1$, $G_m(q) \doteq \prod_{i=1}^m h_i(q)$, $m \in \mathbb{N}$, $p > 0$, then $x(t) - \int_0^t (dQ * x) = e$. Indeed,

$$\int\limits_0^t (dQ * x) = \sum_n \lambda^n \sum_{k+m=n} \int\limits_0^t \left(dQ_k \cdot x_m \left(F^{[k]}(\cdot)\right)\right) = \sum_{n=1}^{\infty} \lambda^n \int\limits_0^t \left(ds^p \cdot x_{n-1}\left(s^q\right)\right)$$

$$= \sum_{n=1}^{\infty} \lambda^n \int\limits_0^t \frac{1}{G_{n-1}(q)} \left(s^q\right)^{p h_{n-1}(q)} ds^p = \sum_{n=1}^{\infty} \lambda^n \frac{1}{G_n(q)} t^{p h_n(q)} = x(t) - e.$$

The series $x(t)$ is uniformly convergent on D for all λ (if $q \ge 1$) and for $|\lambda| < 1/(1-q)$ (if $q < 1$), and the equation $x(t) - \int_0^t (dQ * x) = e$ corresponds to the equation $\tilde{x}(t) - \lambda \int_0^t \tilde{x}\left(s^q\right) ds^p = 1$.

The examples suggest that the equation $x(t) - \int_\alpha^t (dQ * x) = e$ with variable $F(\cdot)$ is solvable in the algebra $\mathrm{C}_F[\lambda]$, and we proceed to a detailed discussion of this topic.

4. EMBEDDING OF EQUATIONS WITH DEVIATING ARGUMENT IN A FAMILY OF F-INTEGRAL EQUATIONS

We fix a series $Q \in \mathrm{C}_F[\lambda] \doteq \mathrm{C}_F(D)[\lambda]$ whose components are of bounded variation. By Theorem 3, for arbitrary $\alpha, t \in D$, there exists an F-integral $(\mathscr{Q}x)(t) \doteq \int_\alpha^t (dQ * x)$ for any series

$x \in \mathrm{C}_F[\lambda]$. One has the inclusion $\mathscr{O}x \in \mathrm{C}_F[\lambda]$ (moreover, the components of the series $\mathscr{O}x$ are of bounded variation); consequently, we can proceed to the study of Eq. (2), where $f \in \mathrm{C}_F[\lambda]$. In extended form, Eq. (2) reads

$$\sum_n \lambda^n x_n(t) - \sum_n \lambda^n \sum_{k+m=n} \int_\alpha^t \left(dQ_k \cdot x_m^{[k]}\right) = \sum_n \lambda^n f_n(t),$$

which is equivalent to the infinite system of equations $x_n(t) - \sum_{k+m=n} \int_\alpha^t \left(dQ_k \cdot x_m^{[k]}\right) = f_n(t)$, or

$$\begin{aligned} &x_0(t) - \int_\alpha^t (dQ_0 \cdot x_0) = f_0(t), \\ &x_n(t) - \int_\alpha^t (dQ_0 \cdot x_n) = f_n(t) + \sum_{k=1}^n \int_\alpha^t \left(dQ_k \cdot x_{n-k}^{[k]}\right), \qquad n \in \mathbb{N}. \end{aligned} \tag{6}$$

If $Q_0 = \text{const}$, then the system is of recursion character [since the integral on the left-hand side in (6) is zero]; but if $Q_0 \neq \text{const}$, then it consists of integral equations. In both cases, the system is uniquely solvable; therefore, Eq. (2) has a unique solution. Throughout the following, we assume that $Q_0 = \text{const}$.

For the special case in which $Q_k(\cdot) = \delta_{k1} q(\cdot)$, $k = 0, 1, \ldots$, the system acquires the form

$$x_0(t) = f_0(t), \qquad x_n(t) = f_n(t) + \int_\alpha^t (dq \cdot x_{n-1}(F(\cdot))), \qquad n \in \mathbb{N}. \tag{7}$$

Suppose that the series $\sum_n \lambda^n x_n$ and $\sum_n \lambda^n f_n$ are convergent (if $|\lambda| < \varepsilon$) in the metric of the space C and one can exchange summation and integration; then from (7) we obtain

$$\sum_n \lambda^n x_n(t) - \int_\alpha^t \left(dq \cdot \sum_{n=1}^\infty \lambda^n x_{n-1}(F(\cdot))\right) = \sum_n \lambda^n f_n(t),$$

or $\tilde{x}(t) - \lambda \int_\alpha^t (dq \cdot \tilde{x}(F(\cdot))) = \tilde{f}(t)$, where $\tilde{x} \doteq \sum_n \lambda^n x_n$ and $\tilde{f} \doteq \sum_n \lambda^n f_n$ are the sums of the series. Therefore, every equation (1) can be embedded in a family of F-integral equations (2).

5. THE CAUCHY SERIES OF AN F-INTEGRAL EQUATION

Let $f = e$ in the F-integral equation (2); i.e., $f_n(t) = \delta_{n0}$, and the coefficients of the series Q are continuous and of bounded variation; moreover, $Q_0 = \text{const}$. By $X(t)$ we denote the solution of this equation. In other words, $X(t) - \int_\alpha^t (dQ * X) = e$. By (6), we have $X_0 = f_0 = 1$. By Lemma 1, the series $X(t)$ is invertible in the algebra $\mathrm{C}_F(D)[\lambda]$; i.e., there exists a series $Y(t)$ such that $X(t) * Y(t) = e = Y(t) * X(t)$.

Definition 2. The *Cauchy series* $C(t, \tau)$ of the F-integral equation (2) is the series in the algebra $\mathrm{C}_F\left(D^2\right)[\lambda]$ given by the relation $C(t, \tau) \doteq X(t) * Y(\tau)$.

Obviously, $C(s, s) = e$, the relation $C(t, s) * C(s, \tau) = C(t, \tau)$ is valid in the algebra $\mathrm{C}_F\left(D^3\right)[\lambda]$, and $C(t, \tau)$ and $C(\tau, t)$ are mutually inverse series in the algebra $\mathrm{C}_F\left(D^2\right)[\lambda]$. The following two properties are less obvious: $C(\alpha, \tau) = Y(\tau)$, and if $\alpha = F(\alpha)$, then $C(t, \alpha) = X(t)$. Indeed, by (6), we have $X_k(\alpha) = \delta_{k0}$; consequently, $Y_m(\alpha) = \delta_{m0}$ and

$$C(t, \tau)|_{t=\alpha} = \sum_n \lambda^n \sum_{k+m=n} X_k(\alpha) Y_m\left(F^{[k]}(\tau)\right) = \sum_n \lambda^n Y_n(\tau) = Y(\tau), \tag{8}$$

$$C(t,\tau)|_{\tau=\alpha} = \sum_n \lambda^n \sum_{k+m=n} X_k(t) Y_m\left(F^{[k]}(\alpha)\right) = \sum_n \lambda^n \sum_{k+m=n} X_k(t) Y_m(\alpha)$$
$$= \sum_n \lambda^n X_n(t) = X(t).$$

Remark 3. Since $F : D \to D$ is a continuous function, it follows that the equation $\alpha = F(\alpha)$ is solvable.

Theorem 4. *The Cauchy problem satisfies the identity* $C(t,\tau) - \int_\tau^t (dQ(s) * C(s,\tau)) = e.$

Indeed, since $X(t) - \int_\alpha^t (dQ * X) = e$, it follows from Lemma 3 that

$$\int_\tau^t (dQ(s) * C(s,\tau)) = \int_\tau^t (dQ(s) * (X(s) * Y(\tau))) = \int_\tau^t (dQ * X) * Y(\tau)$$
$$= \left(\int_\alpha^t (dQ * X) - \int_\alpha^\tau (dQ * X) \right) * Y(\tau)$$
$$= (X(t) - X(\tau)) * Y(\tau) = C(t,\tau) - e.$$

Remark 4. The above-proved formula can be represented in the extended form as

$$C_0(t,\tau) = 1, \qquad C_n(t,\tau) = \sum_{k=1}^n \int_\tau^t \left(dQ_k(s) \cdot C_{n-k}\left(F^{[k]}(s), F^{[k]}(\tau)\right)\right), \qquad n \in \mathbb{N}; \tag{9}$$

therefore, the (continuous) functions $C_n(t,\tau)$ are of bounded variation with respect to the first variable. In other words, for a fixed $\tau \in D$, the section $C_n(\cdot,\tau)$ is of bounded variation; however, in the general case, the cross-section $C_n(t,\cdot)$ with fixed $t \in D$ is not necessarily of bounded variation. For example, if $D = [-1,1]$, $Q_1(t) = t$, $Q_2(t) = 0$, $F(t) = t\cos(\pi/2t)$ for $t \neq 0$, and $F(0) = 0$, then $F : D \to D$ is a continuous function of unbounded variation. We have $C_1(t,\tau) = t - \tau$; therefore, $C_2(t,\tau) = \int_\tau^t (ds \cdot C_1(F(s), F(\tau))) = \int_\tau^t F(s)ds + F(\tau)(\tau - t)$ is a function of unbounded variation with respect to τ.

6. CONVERGENCE OF SOLUTIONS OF F-INTEGRAL EQUATIONS

By $\tilde{\mathrm{C}}[\lambda] \doteq \tilde{\mathrm{C}}\left(D^\ell\right)[\lambda]$ we denote the subspace of $\mathrm{C}[\lambda]$ consisting of series $x \doteq \sum_k \lambda^k x_k$, $x_k \in \mathrm{C}\left(D^\ell\right)$, such that the power series $\sum_k \lambda^k \|x_k\|$ is convergent for small λ. [That is, there exists an $\varepsilon \doteq \varepsilon(x) > 0$ such that the series is convergent if $|\lambda| < \varepsilon$.] This condition is equivalent to the convergence of the series $\sum_k |\lambda|^k \|x_k\|$ in the same neighborhood. In a similar way, we define the subspace $\tilde{\mathrm{CBV}}[\lambda] \doteq \tilde{\mathrm{CBV}}(D)[\lambda]$ that consists of series $x \doteq \sum_k \lambda^k x_k$, $x_k \in \mathrm{CBV}(D)$, such that the series $\sum_k \lambda^k \|x_k\|_{\mathrm{BV}}$ (or $\sum_k |\lambda|^k \|x_k\|_{\mathrm{BV}}$) is convergent for small λ. By $\mathrm{CBV} \doteq \mathrm{CBV}(D)$ we denote the space of continuous functions $x : D \to \mathbb{R}$ of bounded variation, and

$$\|x\| \doteq \max_{(t_1,\ldots,t_\ell)\in D^\ell} |x(t_1,\ldots,t_\ell)| \quad \text{and} \quad \|x\|_{\mathrm{BV}} \doteq |x(a)| + \operatorname*{Var}_D x$$

are the norms in $\mathrm{C}\left(D^\ell\right)$ and $\mathrm{CBV}(D)$, respectively. Since $\|x\| \le \|x\|_{\mathrm{BV}}$ for $\ell = 1$, we have $\tilde{\mathrm{CBV}}(D)[\lambda] \subset \tilde{\mathrm{C}}(D)[\lambda]$. The inequality $\left\|x^{[k]}\right\| \le \|x\|$ is valid for arbitrary $F : D \to D$ and $k = 0, 1, \ldots$

If $x, y \in \tilde{\mathrm{C}}[\lambda]$, then $x * y \in \tilde{\mathrm{C}}[\lambda]$. Indeed, we have

$$\sum_n |\lambda|^n \left\| \sum_{k+m=n} x_k y_m^{[k]} \right\| \le \sum_n |\lambda|^n \sum_{k+m=n} \|x_k\| \|y_m\| = \left(\sum_k |\lambda|^k \|x_k\| \right) \left(\sum_m |\lambda|^m \|y_m\| \right) < \infty$$

for small λ. The inclusions $\gamma x, x+y \in \tilde{\mathrm{C}}[\lambda]$ are obvious ($\gamma \in \mathbb{R}$); therefore, the subset $\tilde{\mathrm{C}}[\lambda]$ is closed under the operations of the algebra $\mathrm{C}_F[\lambda]$ and is an algebra itself. (We denote it by $\tilde{\mathrm{C}}_F[\lambda]$.)

Assertion 1. *If $x \in \tilde{\mathrm{C}}_F[\lambda]$ and $y \in \mathrm{C}_F[\lambda]$ satisfy the condition $x * y = e$, then $y \in \tilde{\mathrm{C}}_F[\lambda]$.*

Proof. By Lemma 1, the coefficient x_0 is invertible in the algebra C.

1. Suppose that $x_0 = 1$. If $x \doteq 1 + \sum_{k=1}^{\infty} \lambda^k x_k$, then, by Lemma 2,

$$y = 1 + \sum_{m=1}^{\infty} \lambda^m \sum_{p_1+\cdots+p_r=m} \prod_{i=1}^{r} \left(-x_{p_i}^{[m-p_1-\cdots-p_i]} \right).$$

Since $x \in \tilde{\mathrm{C}}_F[\lambda]$, it follows that the series

$$1 - \sum_{k=1}^{\infty} |\lambda|^k \|x_k\|, \qquad 1 + \sum_{m=1}^{\infty} |\lambda|^m \sum_{p_1+\cdots+p_r=m} \prod_{i=1}^{r} \|x_{p_i}\|$$

are convergent for small λ (see Remark 1 and [11, p. 210]). One can readily see that the last series dominates the series $1 + \sum_{m=1}^{\infty} |\lambda|^m \left\| \sum_{p_1+\cdots+p_r=m} \prod_{i=1}^{r} \left(-x_{p_i}^{[m-p_1-\cdots-p_i]} \right) \right\|$; therefore, $y \in \tilde{\mathrm{C}}_F[\lambda]$.

2. Let x_0 be an arbitrary invertible function. If $u_k = \delta_{k0} x_0$, $v_m = x_0^{-1} x_m$, $u \doteq \sum_k \lambda^k u_k$, and $v \doteq \sum_m \lambda^m v_m$, then $u * v = \sum_n \lambda^n \sum_{k+m=n} \delta_{k0} x_0 v_m^{[k]} = \sum_n \lambda^n x_0 v_n = x$; consequently, $y = x^{-1} = v^{-1} * u^{-1}$. In this case, the inverse series u^{-1} and v^{-1} exist and belong to the algebra $\tilde{\mathrm{C}}_F[\lambda]$ (this is obvious for the first series; the second series satisfies the condition $v_0 = 1$, and therefore, the assumptions of item 1 are valid for it); consequently, $y \in \tilde{\mathrm{C}}_F[\lambda]$.

Assertion 2. *Let $Q \in \mathrm{C}\tilde{\mathrm{B}}\mathrm{V}[\lambda]$. The operator $\mathscr{Q} : \mathrm{C}[\lambda] \to \mathrm{C}[\lambda]$ given by the formula $(\mathscr{Q}x)(t) \doteq \int_\alpha^t (dQ * x)$ is a mapping of $\tilde{\mathrm{C}}[\lambda]$ into $\tilde{\mathrm{C}}[\lambda]$.*

Indeed, if $x \in \tilde{\mathrm{C}}[\lambda]$, then we have the chains of inequalities

$$\left| \sum_{k+m=n} \int_\alpha^t \left(dQ_k \cdot x_m^{[k]} \right) \right| \le \sum_{k+m=n} \left| \int_\alpha^t \left(dQ_k \cdot x_m^{[k]} \right) \right| \le \sum_{k+m=n} \operatorname*{Var}_D Q_k \cdot \left\| x_m^{[k]} \right\| \le \sum_{k+m=n} \|Q_k\|_{\mathrm{BV}} \|x_m\|,$$

$$\sum_n |\lambda|^n \left\| \sum_{k+m=n} \int_\alpha^t \left(dQ_k \cdot x_m^{[k]} \right) \right\| \le \sum_n |\lambda|^n \sum_{k+m=n} \|Q_k\|_{\mathrm{BV}} \|x_m\| = \left(\sum_k |\lambda|^k \|Q_k\|_{\mathrm{BV}} \right) \left(\sum_m |\lambda|^m \|x_m\| \right)$$

for all $t \in D$. This is a product of convergent series (for small λ); therefore, $\mathscr{Q}x \in \tilde{\mathrm{C}}[\lambda]$.

Theorem 5. *Let* $Q \in \mathrm{C\tilde{B}V}[\lambda]$, $Q_0 = \mathrm{const}$, *and* $f \in \tilde{\mathrm{C}}[\lambda]$ *in Eq.* (2). *If* x *is a solution of this equation, then* $x \in \tilde{\mathrm{C}}[\lambda]$.

Proof. Since $Q_0 = \mathrm{const}$, it follows from (6) that

$$\|x_0\| = \|f_0\|, \qquad \|x_1\| \le \|f_1\| + \mathrm{Var}\, Q_1 \cdot \|x_0\| = \|f_1\| + \mathrm{Var}\, Q_1 \cdot \|f_0\|.$$

Let $n > 1$. Suppose that

$$\|x_m\| \le \sum_{p=0}^{m} a_{m-p} \|f_p\| \tag{10}$$

for all $m < n$, where $a_0 \doteq 1$, $a_q \doteq \sum_{p_1+\dots+p_r=q} \prod_{i=1}^{r} \mathrm{Var}\, Q_{p_i}$, $q \in \mathbb{N}$, and let us prove the estimate (10) for $m = n$. By (6) and (10), we have the chain of inequalities

$$\|x_n\| \le \|f_n\| + \sum_{k=1}^{n} \mathrm{Var}\, Q_k \cdot \|x_{n-k}\| \le \|f_n\| + \sum_{k=1}^{n} \mathrm{Var}\, Q_k \sum_{p=0}^{n-k} a_{n-k-p} \|f_p\|$$
$$= \|f_n\| + \sum_{p=0}^{n-1} c_p \|f_p\|,$$

where $c_{n-1} \doteq \mathrm{Var}\, Q_1 \cdot a_0 = a_1$, and for $p = 0, \dots, n-2$, we have

$$c_p \doteq \sum_{k=1}^{n-p} \mathrm{Var}\, Q_k \cdot a_{n-p-k} = \mathrm{Var}\, Q_{n-p} + \sum_{k=1}^{n-p-1} \sum_{p_1+\dots+p_r=n-p-k} \mathrm{Var}\, Q_k \cdot \prod_{i=1}^{r} \mathrm{Var}\, Q_{p_i}$$
$$= \mathrm{Var}\, Q_{n-p} + \sum_{\substack{k+p_1+\dots+p_r=n-p \\ k<n-p}} \mathrm{Var}\, Q_k \cdot \prod_{i=1}^{r} \mathrm{Var}\, Q_{p_i}.$$

Here double summation have been replaced by joint summation with respect to all variables. By replacing the indices as $\nu_1 = k$, $\nu_2 = p_1$, $\dots$, $\nu_{r+1} = p_r$ and then $q = r+1$, $j = i+1$, we obtain the relation

$$c_p = \mathrm{Var}\, Q_{n-p} + \sum_{\substack{\nu_1+\dots+\nu_{r+1}=n-p \\ \nu_1<n-p}} \prod_{i=0}^{r} \mathrm{Var}\, Q_{\nu_{i+1}}$$
$$= \mathrm{Var}\, Q_{n-p} + \sum_{\substack{\nu_1+\dots+\nu_q=n-p \\ \nu_1<n-p}} \prod_{j=1}^{q} \mathrm{Var}\, Q_{\nu_j} = a_{n-p}.$$

Therefore, the estimate (10) remains valid for $m = n$; consequently,

$$\sum_n |\lambda|^n \|x_n\| \le \sum_n |\lambda|^n \sum_{k=0}^{n} a_{n-k} \|f_k\| = \left(\sum_m |\lambda|^m a_m\right)\left(\sum_k |\lambda|^k \|f_k\|\right)$$
$$= \left(1 - \sum_{k=1}^{\infty} |\lambda|^k \mathrm{Var}\, Q_k\right)^{-1} \left(\sum_k |\lambda|^k \|f_k\|\right) < \infty$$

for small λ. The last equation is valid by virtue of Remark 1. The convergence of the second series in the last product takes place by virtue of the inclusion $f \in \tilde{\mathrm{C}}[\lambda]$, and the convergence of the first series follows from the inclusion $Q \in \mathrm{C\tilde{B}V}[\lambda]$ (since $\sum_{k=1}^{\infty} |\lambda|^k \mathrm{Var}\, Q_k \le \sum_{k=1}^{\infty} |\lambda|^k \|Q_k\|_{\mathrm{BV}} < \infty$) and from the convergence of the inverse series for small λ [11, p. 210].

Remark 5. For Eq. (1), whose generalization is given by Eq. (2) with the kernel Q such that $Q_k(\cdot) = \delta_{k1} q(\cdot)$, the inclusion $f \in \tilde{\mathrm{C}}[\lambda]$ in (2) necessarily implies that $x \in \tilde{\mathrm{C}}[\lambda]$.

Remark 6. Let $Q \in \mathrm{C\tilde{B}V}[\lambda]$. Since $C(t,\tau) = X(t) * Y(\tau)$, $X \in \tilde{\mathrm{C}}(D)[\lambda]$, and, by Assertion 1, $Y \in \tilde{\mathrm{C}}(D)[\lambda]$, we have $C(t,\tau) \in \tilde{\mathrm{C}}\left(D^2\right)[\lambda]$. Moreover, the inclusion $C(\cdot,\tau) \in \mathrm{C\tilde{B}V}[\lambda]$ is valid for a given $\tau \in D$. Indeed, by (9), we have

$$\begin{aligned}\operatorname{Var} C_n(\cdot\,,\tau) &\le \sum_{k=1}^{n} \operatorname{Var} Q_k \cdot \left\| C_{n-k}\left(\cdot, F^{[k]}(\tau)\right)\right\| \le \sum_{k=1}^{n} \operatorname{Var} Q_k \cdot \|C_{n-k}\| \\ &= \sum_{k=0}^{n} \operatorname{Var} Q_k \cdot \|C_{n-k}\|\end{aligned}$$

for $n \in \mathbb{N}$. The estimate remains valid for $n = 0$; consequently, for small λ, we have the chain of inequalities

$$\begin{aligned}\sum_n |\lambda|^n \operatorname{Var} C_n(\cdot,\tau) &\le \sum_n |\lambda|^n \sum_{k=0}^{n} \operatorname{Var} Q_k \cdot \|C_{n-k}\| \\ &= \left(\sum_k |\lambda|^k \operatorname{Var} Q_k\right)\left(\sum_m |\lambda|^m \|C_m\|\right) < \infty,\end{aligned}$$

and since $\|C_n(\cdot,\tau)\|_{\mathrm{BV}} \le \|C_n\| + \operatorname{Var} C_n(\cdot,\tau)$, it follows that $\sum_n |\lambda|^n \|C_n(\cdot,\tau)\|_{\mathrm{BV}} < \infty$; consequently, $C(\cdot,\tau) \in \mathrm{C\tilde{B}V}[\lambda]$. By the example in Remark 4, there exist deviations $F : D \to D$ such that, for fixed $t \in D$, the sections $C_n(t,\cdot)$ are of unbounded variation. Our aim is to describe the class of kernels Q for which $C_n(t,\cdot) \in \mathrm{CBV}(D)$ for all $n \in \mathbb{N}$ and $t \in D$ and consequently, the representation (13) is valid for solutions of F-integral equations.

7. ADDITIONAL ASSERTIONS ABOUT F-INTEGRALS

In the preceding sections, we have considered a kernel Q of Eq. (2) such that $Q_k \in \mathrm{CBV} \doteq \mathrm{CBV}(D)$, i.e., all Q_k are continuous functions of bounded variation. We fix a continuous function $F : D \to D$, and by $\mathrm{CBF} \doteq \mathrm{CBF}(D)$ we denote the subspace of CBV consisting of $x : D \to \mathbb{R}$ such that $x^{[k]} = x\left(F^{[k]}(\cdot)\right) \in \mathrm{CBV}$ for all $k = 0, 1, \ldots$ One can readily show that CBF = CBV for any continuous piecewise monotone function $F : D \to D$. (A continuous function $z : [a,b] \to \mathbb{R}$ is said to be *piecewise monotone* if there exists a partition $a = \tau_0 < \tau_1 < \cdots < \tau_n = b$ such that the restriction $z : [\tau_{k-1}, \tau_k] \to \mathbb{R}$ is a monotone function for all $k = 1, \ldots, n$.)

Lemma 4. *If* $u, u(F(\cdot)) \in \mathrm{CBV}$, $v \in \mathrm{C}$, *and* $[\alpha,\beta] \subseteq D$, *then*

$$\int_{F(\alpha)}^{F(\beta)} (du \cdot v) = \int_{\alpha}^{\beta} (du(F(\cdot)) \cdot v(F(\cdot))) \quad \text{and} \quad \int_{F(\alpha)}^{F(\beta)} (u \cdot dv) = \int_{\alpha}^{\beta} (u(F(\cdot)) \cdot dv(F(\cdot))).$$

These formulas are called the change of variables formulas for the Riemann–Stieltjes integral, and their proof can be performed in a standard way on the basis of the comparison of integral sums.

Lemma 5. *If* $u \in \mathrm{CBF}$, $v \in \mathrm{C}$, $\alpha \in D$, *and* $w(t) \doteq \int_\alpha^t (du \cdot v)$, *then* $w \in \mathrm{CBF}$.

Proof. The inclusion $w \in \mathrm{CBV}$ is well known [10, p. 216]. Without loss of generality, we assume that $\alpha < t$. For any $k \in \mathbb{N}$, we have the inclusion

$$u^{[k]} = u\left(F^{[k]}(\cdot)\right) \in \mathrm{CBV};$$

consequently, by Lemma 4, we have the chain of relations

$$\sum_{m=1}^{n}\left|w\left(F^{[k]}\left(s_m\right)\right)-w\left(F^{[k]}\left(s_{m-1}\right)\right)\right| = \sum_{m=1}^{n}\left|\int\limits_{F^{[k]}(s_{m-1})}^{F^{[k]}(s_m)}(du\cdot v)\right|$$
$$=\sum_{m=1}^{n}\left|\int\limits_{s_{m-1}}^{s_m}\left(du^{[k]}\cdot v^{[k]}\right)\right| \le \operatorname*{Var}_D u^{[k]}\cdot\|v\|$$

for an arbitrary partition $\alpha = s_0 < s_1 < \cdots < s_n = t$.

Lemma 6. *Let $u, v, w \in \mathrm{C}[\lambda]$ and $\alpha, \beta \in D$. Then the following assertions are valid.*

1. *If $u_n \in \mathrm{CBV}$ for all $n = 0, 1, \ldots,$ then* $\int_\alpha^\beta\left(d\int_\alpha^t(du*v)*w(t)\right)=\int_\alpha^\beta(du*(v*w))$.
2. *If $v_n \in \mathrm{CBF}$ for all $n = 0, 1, \ldots,$ then* $\int_\alpha^\beta\left(u(t)*d\int_\alpha^t(dv*w)\right)=\int_\alpha^\beta\left(d\int_\alpha^t(u*dv)*w(t)\right)$.
3. *If $w_n \in \mathrm{CBF}$ for all $n = 0, 1, \ldots,$ then* $\int_\alpha^\beta\left(u(t)*d\int_\alpha^t(v*dw)\right)=\int_\alpha^\beta((u*v)*dw)$.

Proof. By Lemma 5, all F-integrals exist. We prove only the third formula, and the proof of the first two ones can be performed in a symmetric way. The expressions $\int_\alpha^t(v*dw)$ and $u*v$ are equal to $\sum_i \lambda^i \sum_{m+n=i}\int_\alpha^t\left(v_m\cdot dw_n^{[m]}\right)$ and $\sum_i \lambda^i\sum_{k+m=i}u_k v_m^{[k]}$, respectively; consequently, by virtue of the second formula in Lemma 4, we have the chain of relations

$$\int\limits_\alpha^\beta\left(u(t)*d\int\limits_\alpha^t(v*dw)\right)$$
$$=\sum_j\lambda^j\sum_{k+i=j}\int\limits_\alpha^\beta\left(u_k(t)\cdot d\sum_{m+n=i}\left[\int\limits_{F^{[k]}(\alpha)}^{F^{[k]}(t)}\left(v_m\cdot dw_n^{[m]}\right)+\int\limits_\alpha^{F^{[k]}(\alpha)}\left(v_m\cdot dw_n^{[m]}\right)\right]\right)$$
$$=\sum_j\lambda^j\sum_{k+m+n=j}\int\limits_\alpha^\beta\left(u_k(t)\cdot d\int\limits_\alpha^t\left(v_m^{[k]}\cdot dw_n^{[k+m]}\right)\right)$$
$$=\sum_j\lambda^j\sum_{k+m+n=j}\int\limits_\alpha^\beta\left(u_kv_m^{[k]}\cdot dw_n^{[k+m]}\right)=\sum_j\lambda^j\sum_{i+n=j}\int\limits_\alpha^\beta\left(\sum_{k+m=i}u_kv_m^{[k]}\cdot dw_n^{[i]}\right)$$
$$=\int\limits_\alpha^\beta((u*v)*dw).$$

8. THE ADJOINT F-INTEGRAL EQUATION

Let $Q_k \in \mathrm{CBF}$ for all $k \in \mathbb{N}$ and $Q_0 = \mathrm{const}$ in Eq. (2). If $C(t,\tau)$ is the Cauchy series of this equation, then Theorem 4, as well as Theorem 6 below, is valid. Anticipating its proof, we justify the relations

$$\int\limits_\alpha^\beta(dX*Y)=\int\limits_\alpha^\beta dQ,\qquad \int\limits_\alpha^\beta(X*dY)=-\int\limits_\alpha^\beta dQ, \tag{11}$$

where $\alpha, \beta \in D$; moreover, $X_n, Y_n \in \mathrm{CBF}$ for all $n = 0, 1, \ldots$ Recall that $X(t)$ is a solution of the equation $X(t)-\int_\alpha^t(dQ*X)=e$, $Y(t)=X^{-1}(t)$, and $X(t)*Y(\tau)=C(t,\tau)$. Therefore, by virtue of (6) and Lemma 5, the inclusions $Q_k \in \mathrm{CBF}$ imply that $X_n \in \mathrm{CBF}$; moreover, $X_0 = 1$, and the inclusion $Y_n \in \mathrm{CBF}$ follows from the relation $X_0 Y_n + \sum_{k=1}^{n} X_k Y_{n-k}^{[k]} = \delta_{n0}$ occurring in the proof of Lemma 1. By virtue of the first formula in Lemma 6, we have the chain of relations

$$\int\limits_{\alpha}^{\beta} (dX * Y) = \int\limits_{\alpha}^{\beta} \left(d \int\limits_{\alpha}^{s} (dQ * X) * Y(s) \right) = \int\limits_{\alpha}^{\beta} (dQ * (X * Y)) = \int\limits_{\alpha}^{\beta} dQ,$$

and the second formula (11) follows from (5) and the identity $X(t) * Y(t) = e$.

Theorem 6. *If the kernel Q in Eq.* (2) *satisfies the conditions $Q_k \in \mathrm{CBF}$ for all $k \in \mathbb{N}$ and $Q_0 = \mathrm{const}$, then the Cauchy series satisfies the identity $C(t,\tau) - \int_{\tau}^{t} (C(t,s) * dQ(s)) = e$.*

Proof. By virtue of (11) and the third formula in Lemma 6, we have

$$\int\limits_{\tau}^{t} (C(t,s) * dQ(s)) = \int\limits_{\tau}^{t} \left(C(t,s) * d_s \int\limits_{\tau}^{s} dQ \right) = -\int\limits_{\tau}^{t} \left(C(t,s) * d_s \int\limits_{\tau}^{s} (X * dY) \right)$$
$$= -\int\limits_{\tau}^{t} ((C(t,s) * X(s)) * dY(s)) = -\int\limits_{\tau}^{t} (X(t) * dY(s))$$
$$= -X(t) * Y(t) + X(t) * Y(\tau) = -e + C(t,\tau).$$

Remark 7. By (8), we have $C(\alpha,\tau) = Y(\tau)$; therefore, if t in the identity in Theorem 6 is replaced by α, then we obtain the identity $Y(\tau) + \int_{\alpha}^{\tau} (Y * dQ) = e$. Therefore, one can claim that if $Q_k \in \mathrm{CBF}$, then the equations

$$x(t) - \int\limits_{\alpha}^{t} (dQ * x) = f(t), \qquad y(\tau) + \int\limits_{\alpha}^{\tau} (y * dQ) = g(\tau) \tag{12}$$

are *adjoint* or form a *pair of adjoint problems* (here $f, g \in \mathrm{C}[\lambda]$). In favor of this terminology, we provide the following reasoning. By repeating the considerations in Section 4 for the second equation (12) with $F(\xi) = \xi$ and $Q_k(t) = \delta_{k1} q(t)$ [see system (7)], we find that the pair (12) corresponds to the equations $\tilde{x}(t) - \lambda \int_{\alpha}^{t} dq \cdot \tilde{x} = \tilde{f}(t)$ and $\tilde{y}(\tau) + \lambda \int_{\alpha}^{\tau} \tilde{y}\, dq = \tilde{g}(\tau)$; and if, in addition, q, $\tilde{f}$, and $\tilde{g}$ are differentiable functions, then we obtain the ordinary differential equations

$$\dot{\tilde{x}}(t) - \lambda \dot{q}(t) \tilde{x}(t) = \dot{\tilde{f}}(t), \qquad \dot{\tilde{y}}(\tau) + \lambda \tilde{y}(\tau) \dot{q}(\tau) = \dot{\tilde{g}}(\tau)$$

with the adjoint operators (one can readily see that similar considerations are valid in the matrix case as well).

9. REPRESENTATION OF SOLUTIONS OF ADJOINT F-INTEGRAL EQUATIONS

Theorem 7. *If the kernel Q in Eq.* (12) *satisfies the conditions $Q_k \in \mathrm{CBF}$, $k \in \mathbb{N}$, and $Q_0 = \mathrm{const}$, then the* (*unique*) *solution of the first equation with $\alpha = F(\alpha)$ can be represented in the form*

$$x(t) = f(t) - \int\limits_{\alpha}^{t} (d_s C(t,s) * f(s)) \quad \text{or} \quad x(t) = C(t,\alpha) * f(\alpha) + \int\limits_{\alpha}^{t} (C(t,s) * df(s)), \tag{13}$$

and the (unique) solution of the second equation can be represented as

$$y(\tau) = g(\tau) - \int_{\alpha}^{\tau} (g(s) * d_s C(s,\tau)) \quad \text{or} \quad y(\tau) = g(\alpha) * C(\alpha,\tau) + \int_{\alpha}^{\tau} (dg(s) * C(s,\tau)) \tag{14}$$

for any $\alpha \in D$; the F-multiplication in the second formulas in (13) *and* (14) *is performed in the algebra* $\mathrm{C}_F(D^2)[\lambda]$.

Proof. By Theorem 6, we have the infinite system of relations

$$C_0(t,\tau) = 1, \qquad C_n(t,\tau) = \sum_{m=1}^{n} \int_{\tau}^{t} \left(C_{n-m}(t,s) \cdot dQ_m\left(F^{[n-m]}(s)\right)\right), \quad n \in \mathbb{N};$$

therefore, for a fixed $t \in D$, the coefficients $C_n(t,\cdot)$ are of bounded variation; consequently, the F-integrals occurring in (13) exist. The existence of the F-integrals occurring in (14) is justified by Remark 4. The existence and uniqueness of the solution of the first equation in (12) have been discussed in comments to system (6), and the existence and uniqueness of the solution of the second equation (12) take place by virtue of a similar argument.

Let us prove the first formula in (14) [the second one follows from it in view of (5)]. By substituting the right-hand side of this formula into the F-integral in the second equation in (12), we obtain the relation $\int_\alpha^\tau (y * dQ) = \int_\alpha^\tau (g * dQ) + \sigma$, where

$$\sigma \doteq -\int_{\alpha}^{\tau} \left(\int_{\alpha}^{s} (g(\xi) * d_\xi C(\xi,s)) * dQ(s)\right) = -\int_{\alpha}^{\tau} \left(\int_{\alpha}^{s} (g * dX) * Y(s) * dQ(s)\right).$$

We have used the relation $\int_E (u(s) * d_s(v(s) * w(\tau))) = \int_E (u * dv) * w(\tau)$, which is valid by Lemma 3 and formula (5). The third formula in Lemma 6, together with Remark 7, implies that

$$\sigma = -\int_{\alpha}^{\tau} \left(\int_{\alpha}^{s} (g * dX) * d\int_{\alpha}^{s} (Y * dQ)\right) = \int_{\alpha}^{\tau} \left(\int_{\alpha}^{s} (g * dX) * dY(s)\right).$$

By the first formula in (11), the second formula in Lemma 6, and (5), we have the chain of relations

$$\int_{\alpha}^{\tau} (g * dQ) = \int_{\alpha}^{\tau} \left(g(s) * d\int_{\alpha}^{s} (dX * Y)\right) = \int_{\alpha}^{\tau} \left(d\int_{\alpha}^{s} (g * dX) * Y(s)\right),$$

$$\int_{\alpha}^{\tau} (y * dQ) = \left.\int_{\alpha}^{s} (g * dX) * Y(s)\right|_{\alpha}^{\tau} = \int_{\alpha}^{\tau} (g * dX) * Y(\tau)$$

$$= \int_{\alpha}^{\tau} (g(s) * d_s C(s,\tau)) = g(\tau) - y(\tau).$$

Anticipating the proof of formulas (13), we justify the relation

$$\int_{\alpha}^{t} (d_s C(t,s) * f(s)) = X(t) * \int_{\alpha}^{t} (dY * f). \tag{15}$$

By Remark 2, a series independent of the integration variable and written on the left cannot in general be transferred outside the F-integral; however, under the assumptions of the theorem, we have the chain of relations

$$\int_{\alpha}^{t} (d_sC(t,s) * f(s)) = \sum_j \lambda^j \sum_{i+n=j} \int_{\alpha}^{t} \left(d_sC_i(t,s) \cdot f_n^{[i]}(s)\right)$$
$$= \sum_j \lambda^j \sum_{i+n=j} \int_{\alpha}^{t} \left(d_s \sum_{k+m=i} X_k(t) Y_m^{[k]}(s) \cdot f_n^{[i]}(s)\right)$$
$$= \sum_j \lambda^j \sum_{k+m+n=j} X_k(t) \int_{\alpha}^{t} \left(dY_m^{[k]} \cdot f_n^{[k+m]}\right).$$

By Lemma 4, the condition $\alpha = F(\alpha)$, and the relation $\int_{\alpha}^{t}(dY * f) = \sum_i \lambda^i \sum_{m+n=i} \int_{\alpha}^{t} \left(dY_m \cdot f_n^{[m]}\right)$, we have

$$\int_{\alpha}^{t} (d_sC(t,s) * f(s)) = \sum_j \lambda^j \sum_{k+m+n=j} X_k(t) \int_{F^{[k]}(\alpha)}^{F^{[k]}(t)} \left(dY_m \cdot f_n^{[m]}\right)$$
$$= \sum_j \lambda^j \sum_{k+i=j} X_k(t) \sum_{m+n=i} \int_{\alpha}^{F^{[k]}(t)} \left(dY_m \cdot f_n^{[m]}\right) = X(t) * \int_{\alpha}^{t} (dY * f).$$

By substituting the right-hand side of the first formula in (13) into the F-integral in the first equation in (12), we obtain $\int_{\alpha}^{t}(dQ * x) = \int_{\alpha}^{t}(dQ * f) + \sigma$, where

$$\sigma \doteq -\int_{\alpha}^{t} \left(dQ(s) * \int_{\alpha}^{s} (d_\xi C(s,\xi) * f(\xi))\right) = -\int_{\alpha}^{t} \left(dQ(s) * X(s) * \int_{\alpha}^{s} (dY * f).\right)$$

[We have used formula (15).] It follows from the first formula in Lemma 6 that

$$\sigma = -\int_{\alpha}^{t} \left(d\int_{\alpha}^{s} (dQ * X) * \int_{\alpha}^{s} (dY * f)\right) = -\int_{\alpha}^{t} \left(dX(s) * \int_{\alpha}^{s} (dY * f)\right).$$

By the second formula in (11), the second formula in Lemma (6), and relation (5), we have the chain of relations

$$\int_{\alpha}^{t} (dQ * f) = -\int_{\alpha}^{t} \left(d\int_{\alpha}^{s} (X * dY) * f(s)\right) = -\int_{\alpha}^{t} \left(X(s) * d\int_{\alpha}^{s} (dY * f)\right),$$
$$\int_{\alpha}^{t} (dQ * x) = -X(s) * \int_{\alpha}^{s} (dY * f)\bigg|_{\alpha}^{t} = -X(t) * \int_{\alpha}^{t} (dY * f) = -\int_{\alpha}^{t} (d_sC(t,s) * f(s)).$$

The penultimate relation is valid by virtue of the condition $\alpha = F(\alpha)$, and the last relation holds by virtue of (15). Therefore, $\int_{\alpha}^{t}(dQ * x) = x(t) - f(t)$. The second formula in (13) is a consequence of the first one.

Summarizing, we note that the schemes of the classical theory of linear differential equations can be transferred to Eq. (2) in full extent. The variable λ plays the role of a spectral parameter in our constructions.

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