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On the Limiting Ellipticity of Galaxies Formed by Dissipationless Collapse

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Summary. Using the model of a homogeneous oblate spheroid that has a global anisotropy of velocity dispersion we show that the limiting ellipticity of a galaxy formed by dissipationless collapse is equal to $\varepsilon_{\max} = 0.8321$. This is considerably greater than the analogous value $\varepsilon_{\max} = 0.7091$ obtained by Thuan and Gott (1975) who used the model of a homogeneous spheroid with isotropic "pressure". However, neither of these models proves that the real galaxies were formed in the course of dissipationless collapse, due to the fact that both models are inadequate to explain the small rotation of elliptical galaxies.

Key words: formation of galaxies — flattening of galaxies

1. Introduction

During recent years a hypothesis has been rather widely spread that elliptical galaxies were formed as a result of dissipationless collapse of progalactic perturbations, whereas the formation of lenticular and spiral galaxies was accompanied by appreciable dissipation of energy. This belief became plausible after Thuan and Gott (1975) showed, using as a model the homogeneous Maclaurin spheroid,[1] that an initially spherical protogalaxy consisting only of stars does relax in the course of dissipationless collapse to an equilibrium configuration, the ellipticity ($\varepsilon = 1 - c/a$) of which does not exceed a value $\varepsilon_{\max} = 0.70905$, resembling the maximum flattening of elliptical galaxies of the type E7. However, as will be shown in this paper, the value $\varepsilon_{\max}$ depends appreciably on the assumption adopted by Thuan and Gott that the resulting equilibrium configuration is the Maclaurin spheroid, i.e. it has an isotropic velocity dispersion. Moreover, a spheroid with such an isotropic "pressure" cannot serve as a model of elliptical galaxies since their flattening, as it is known at present, is too large to be explained by rotation and is conditioned by a global velocity anisotropy (Binney, 1976). We shall see below that the limiting ellipticity reaches a greater value, $\varepsilon_{\max} = 0.8321$, in a more general case of an equilibrium homogeneous spheroid having a global anisotropy in the velocity dispersion.

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[1] Strictly speaking, the model by Thuan and Gott differs from the classical Maclaurin spheroid in respect that they take for equilibrium state $E_{eq} = \frac{1}{2}W_{eq}$, whereas for the Maclaurin spheroid $E_{eq} = W_{eq}(1 - \theta_{eq})$. Here, E is the total energy, W is the gravitational energy, θ is the ratio of rotational energy $E_{\rm rot}$ to $|W|$ and changes from 0 to $\frac{1}{2}$ when the ellipticity ε changes from 0 to 1. However, the ratio $E_{\rm rot}/|W|$ being a function of ε is the same for both the models. Therefore, for the sake of brevity, we shall use in what follows the term "Maclaurin spheroid" for the spheroid with isotropic pressure considered by Thuan and Gott.

2. Homogeneous Oblate Spheroid with Anisotropic Velocity Dispersion

To consider the case of a global anisotropy in the velocity dispersion we shall restrict ourselves to the homogeneous model of an oblate spheroid of the density ρ_0 consisting of stars with equal masses. The anisotropic distribution function $f(\boldsymbol{r}, \boldsymbol{v})$, depending on the three integrals of motion, obeys an equation (Polyatshenko and Fridman, 1976):

$$\rho_0\eta\left(1-\frac{r^2}{a^2}-\frac{z^2}{c^2}\right)=\frac{A}{4}\int\int\int\frac{f(m,n,T_z)dmdndT_z}{(2T_z-C^2z^2)^{1/2}[(m-r^2)(r^2-n)]^{1/2}}. \quad (1)$$

Here, a and c are the semi-major and semi-minor axes of the spheroid;

$$A^2 = 2\pi G\rho_0\left(\frac{(1-e^2)^{1/2}}{e^3}\arcsin e - \frac{1-e^2}{e^2}\right),$$

$$C^2 = \frac{4\pi G\rho_0}{e^2}\left(1-(1-e^2)^{1/2}\frac{\arcsin e}{e}\right),$$

are coefficients in the expression for the gravitational potential $\varphi = \text{const} + \frac{1}{2}A^2r^2 + \frac{1}{2}C^2z^2$ in a cylindrical coordinate system; e is an eccentricity; m and n describe the geometry of star orbits; T_z is the z-component of the integral of energy $T = (v^2/2) + \varphi(r, z)$; $\eta(x) = 0$ at $x < 0$ and $\eta(x) = 1$ at $x \geqq 0$.

It can be shown that for the stationary spheroid the solution of Eq. (1) is

$$f(m, n, T_z) = \frac{8\rho_0}{\pi^2}\frac{a}{ACc}\frac{\sqrt{m}}{m-n}\,\delta[m - a^2(1 - 2T_z/C^2c^2)], \quad (2)$$

where $\delta[\ldots]$ is the δ-function. The components of velocity dispersion obtained with the help of (2) in the Appendix are (bar means an averaging over the velocity space):

$$\overline{v_r^2} = \frac{a^2A^2}{4}\left(1 - \frac{r^2}{a^2} - \frac{z^2}{c^2}\right)$$
$$\overline{v_z^2} = \frac{c^2C^2}{2}\left(1 - \frac{r^2}{a^2} - \frac{z^2}{c^2}\right) \tag{3}$$
$$\sigma_\varphi^2 = \overline{v_\varphi^2} - \bar{v}_\varphi^2.$$

The transversal component of the velocity dispersion contains a term

$$\overline{v_\varphi^2} = \frac{a^2A^2}{2}\left[1 - \frac{1}{2}\left(1 - \frac{r^2}{a^2} + \frac{z^2}{c^2}\right)\right] \tag{4}$$

and square of the rotational velocity

$$\bar{v}_\varphi = \frac{4aA}{\pi^2}(1 - z^2/c^2)^{1/2} E\left\{\frac{\pi}{2}; [(1 - r^2/a^2 - z^2/c^2)/(1 - z^2/c^2)]^{1/2}\right\} \tag{5}$$

where $E(\ldots)$ is the complete elliptic integral of the second kind. The rotation of the spheroid, being differential, depends both on r and z; at the given r the rotational velocity increases to the plane $z = 0$. The angular velocity increases from the centre to the boundary where it reaches a constant (not depending on both r and z) value $\omega = 2A/\pi$. The boundary surface rotates as a hard egg-shell; all the motions on the boundary disturbing its hardness are absent. As is seen from (3), both $\overline{v_r^2}$ and $\overline{v_z^2}$ decrease from the centre to the boundary of the spheroid, being constant on surfaces $(r^2/a^2) + (z^2/c^2) = \alpha$ ($\alpha \leqq 1$), and when $\alpha = 1$ (the boundary) they come to zero. Note that the uniform rotation of the boundary surface does not mean that it consists of the same stars; in fact, star orbits only touch this boundary, and the condition of such a contact is a reason for the presence of the δ-function in (2).

Averaging (3) and $\overline{v_\varphi^2}$ over the spheroid's volume we get the components of the energy of chaotic motions as well as the rotational energy. The latter may be presented in the form

$$\frac{E_{\rm rot}}{|W|} \cong 0{,}30\left(\frac{1}{e^2} - \frac{(1 - e^2)^{1/2}}{e \cdot \arcsin e}\right). \tag{6}$$

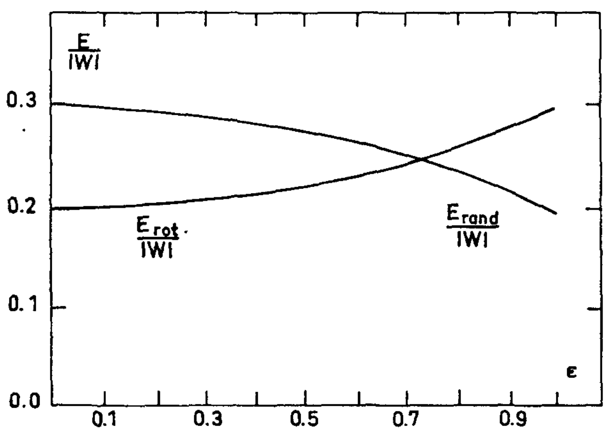

Fig. 1. $E_{\rm rot}/|W|$ and $E_{\rm rand}/|W|$ as functions of ellipticity $\varepsilon = 1 - c/a$

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The energy of chaotic motions, $E_{\rm rand}$, and that of rotation, $E_{\rm rot}$, divided to the modulus of gravitational energy $|W|$, are shown in Fig. 1 as a function of the ellipticity ε of the spheroid. At ellipticities $0 \leqq \varepsilon \leqq 0.7$ an inequality $E_{\rm rand} > E_{\rm rot}$ holds, and at $\varepsilon \gtrsim 0.7$ the opposite inequality takes place. It is interesting that at $\varepsilon = 0$ the rotational energy differs from zero, and this fact demonstrates clearly the specific properties of a homogeneous spheroid with anisotropic pressure. Recall that the rotational energy comes to zero at $\varepsilon = 0$ for both liquid figures of equilibrium and collisionless star systems with isotropic pressure; the rotational velocities are non-zero at $\varepsilon = 0$ only in the curious case of a sphere, the top and bottom halfs of which rotate in opposite directions (Lynden-Bell, 1960), whereas the total angular momentum of the sphere is zero.

As can be seen from Fig. 1, in the model considered $0.2 \leqq E_{\rm rot}/|W| \leqq 0.3$, i.e. the stability condition $E_{\rm rot}/|W| \lesssim 0.14$ suggested by Ostriker and Peebles (1973), is not fulfilled. Therefore this model is unstable to a bar formation unless a spherical component, in addition to a flat one, is present; lenticular and many spiral galaxies contain, as it is well known, such a component. The presence of a centrally-condensed bulge prevents the bar formation as well (Berman and Mark, 1978). This is the reason why we do not believe that the violation of the stability condition for the very simplified (homogeneous!) model considered here is its serious defect.

3. Limiting Ellipticity of a Homogeneous Spheroid with Anisotropic Pressure

To obtain the limiting ellipticity of a homogeneous spheroid with a global anisotropy in the velocity dispersion we assume, after Thuan and Gott (1975), that the spheroid was formed in the course of dissipationless collapse from a spherical homogeneous protogalaxy with initial radius a_i and angular velocity of uniform rotation ω_i (initial random velocities are assumed to be absent).

The absence of any dissipation in the course of collapse does mean, firstly, that the total energy in the initial state

$$E_i = W_i(1 - \theta_i) = -\frac{3M^2G}{5a_i}(1 - \theta_i), \tag{7}$$

where $\theta_i = (E_{\rm rot}/|W|)_i$ is the initial ratio of rotational and potential energies, is equal to the total energy of the resulting equilibrium spheroid

$$E_{eq} = \frac{1}{2}W_{eq} = -\frac{3}{10}\frac{M^2G}{a_{eq}}\frac{\arcsin e}{e}. \tag{8}$$

In other words,

$$(1 - \theta_i) = \frac{1}{2}\frac{\arcsin e}{e}\cdot\frac{a_i}{a_{eq}}. \tag{9}$$

Secondly, the angular momentum of a protogalaxy is conserved, i.e.

$$a_i^2 \cdot \omega_i = a_{eq}^2 \cdot \omega_{eq}, \tag{10}$$

where

$$\omega_{eq} = \frac{5}{2a_{eq}^2} < r\bar{v}_\varphi > \equiv \frac{5}{2a_{eq}^2}\frac{1}{V}\int\int\int r\bar{v}_\varphi dV \tag{11}$$

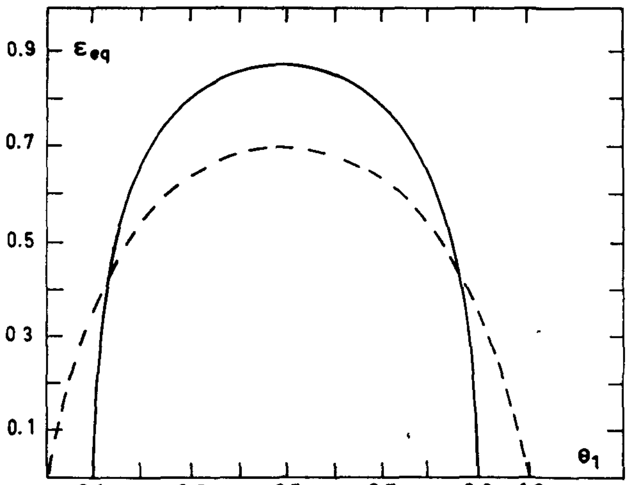

Fig. 2. ε_{eq} as a function of θ_i for spheroids with isotropic "pressure" (dashed line, after Thuan and Gott, 1975) and with anisotropy of velocity dispersion (solid line)

is the angular velocity of the equilibrium spheroid averaged over its volume.

The value ω_{eq}/ω_i may be found using the ratio $E_{\mathrm{rot}\,eq}/E_{\mathrm{rot}\,i}$[2]:

$$\frac{E_{\mathrm{rot}\,eq}}{E_{\mathrm{rot}\,i}} = \frac{a_{eq}^2 \cdot \bar{\omega}_{eq}^2}{a_i^2 \cdot \omega_i^2} = 0{,}30\left(\frac{1}{e^2} - \frac{(1-e^2)^{1/2}}{e \cdot \arcsin e}\right)\frac{\arcsin e}{e}\,\frac{1}{\theta_i}\,\frac{a_i}{a_{eq}}, \quad (12)$$

where

$$\bar{\omega}_{eq}^2 = (5/2a_{eq}^2)\cdot\langle \bar{v}_\varphi^2 \rangle. \quad (13)$$

Eliminating ω_i in (12) with the help of (10), and a_{eq}/a_i with the help of (9), we get

$$\theta_i(1-\theta_i) = 0{,}1362\left(\frac{\arcsin e}{e}\right)^2 \cdot \left[\frac{1}{e^2} - \frac{(1-e^2)^{1/2}}{e \cdot \arcsin e}\right]. \quad (14)$$

The solution of Eq. (14) relative to $\varepsilon = 1 - (1 - e^2)^{1/2}$ is shown in Fig. 2 together with the solution of the corresponding Eq. (7) in Thuan and Gott (1975) which differs from (14) by the right hand side.

There are no equilibrium oblate spheroids with anisotropic velocity dispersion formed from protogalaxies having $\theta_i \lesssim 0.1010$ or $\theta_i > 0.8990$. These limiting values of θ_i correspond to $\varepsilon_{eq} = 0$. When θ_i increases starting from 0.1, the equilibrium ellipticity ε_{eq} increases as well, and at $\theta_i = 1/2$ it reaches a maximum value $\varepsilon_{\max}$ after which ε_{eq} decreases again and turns into zero at $\theta_i \cong 0.8990$.

The physical reason for the existence of the maximum ellipticity for an equilibrium homogeneous spheroid (irrespective of whether its pressure is isotropic or anisotropic) is as follows. At fixed M, a_i ,and the character of initial rotation, the value $\theta_i = (E_{\mathrm{rot}}/|W|)_i$ determines both the angular momentum J_i and the total energy E_i of a protogalaxy. Then the ellipticity of a spheroid in its equilibrium state, $\varepsilon_{eq} = 1 - c_{eq}/a_{eq}$, is determined by the conserved values of J_i and E_i. The semi-axes a_{eq}/a_i and c_{eq}/c_i, normalized to their initial values, depend on θ_i in different manners: the change of a_{eq}/a_i with θ_i is determined by conservation of both the angular momentum and the energy, whereas the change of c_{eq}/c_i with θ_i is determined only by conservation of the total energy. As a result, ε_{eq} as a function of θ_i has a maximum at some value of θ_i.

As for the maximum ellipticity for the homogeneous spheroid with anisotropic pressure, we have the following picture. At the minimum value of $\theta_i = 0.1010$, one obtains $a_{eq}/a_i = c_{eq}/c_i = 0.56$, $\varepsilon_{eq} = 0$, and the density of the equilibrium sphere has the maximum value. With the increasing of θ_i in the range $0.1010 \leqq \theta_i \leqq 1/2$ the value of a_{eq}/a_i is increased (from 0.56 up to 1.42), and c_{eq}/c_i is decreased to the minimum value (0.24 at $\theta_i = 0.42$), after which it begins to increase. As a result, the value of ε_{eq} is increased from zero up to $\varepsilon_{\max} = 0.8321$ which is reached at $\theta_i = 1/2$. This value is the maximum ellipticity because at $\theta_i > 1/2$ the ellipticity decreases since the ratio c_{eq}/c_i increases much faster than a_{eq}/a_i due to the fact that $E_{\mathrm{rot}\,i} \sim \omega_i^2$ whereas $J_i \sim \omega_i$.

It is very interesting that the maximum ellipticity, $\varepsilon_{\max} = 0.8321$, of the equilibrium homogeneous spheroids with anisotropic pressure is appreciably greater than the corresponding value, $\varepsilon_{\max} = 0.7091$, obtained by Thuan and Gott (1975) for the Maclaurin spheroids.

4. Discussion

Is there any correspondence between the considered model of a homogeneous spheroid with anisotropic pressure and properties of galaxies?

Using observed amplitudes of both galaxy rotational velocities v_{rot} and rms chaotic velocity σ, it is possible to evaluate a ratio of energies for rotational and chaotic motions $E_{\mathrm{rot}}/E_{\mathrm{rand}} \approx (v_{\mathrm{rot}}/\sigma)^2$. For elliptical galaxies this ratio is small enough being about 0.1–0.2 (Illingworth, 1977). As for the model considered, the ratio

$$\frac{E_{\mathrm{rot}}}{E_{\mathrm{rand}}} = \left[1{,}67\left(\frac{1}{e^2} - \frac{(1-e^2)^{1/2}}{e \cdot \arcsin e}\right) - 1\right]^{-1} \quad (15)$$

ranges between 2/3 for $\varepsilon = 0$ (a sphere) and 3/2 for $\varepsilon = 1$ (a disk). Therefore this model does not apply at any ε to elliptical galaxies.

At the same time, the dynamical properties of this model resemble rather well some properties of lenticular galaxies. For example, for NGC 128 the ratio $(v_{\mathrm{rot}}/\sigma)^2 \cong 1.2$ and the ellipticity (using the most flattened isophotes) $\varepsilon_m = 0.77$ (Bertola and Capaccioli, 1977a); for NGC 4672 we have $(v_{\mathrm{rot}}/\sigma)^2 \cong 1.1$ and $\varepsilon_m = 0.85$ (Bertola and Capaccioli, 1977b). The theory gives [see Eq. (15)] $E_{\mathrm{rot}}/E_{\mathrm{rand}} \cong 1.14$ and 1.19 at $\varepsilon = 0.77$ and 0.85, respectively. The agreement between the theory and observations is rather good. However, the accuracy of the data available should not be overestimated since errors in the course of measurements of σ are still rather large. Nevertheless, the similarity of dynamics of SO galaxies with the dynamics of the model considered, e.g. with the predominance of rotational velocities upon chaotic ones ($E_{\mathrm{rot}} > E_{\mathrm{rand}}$) at ellipticities $\varepsilon \gtrsim 0.7$, is worth attention. It is interesting that the ellipticity $\varepsilon \cong 0.7$, at which elliptical galaxies "disappear", is near the limit of dynamical stability for Maclaurin spheroids with isotropic pressure. The flattening of flatter systems (SO and Sa-Sc galaxies) is, on the average, of about $\varepsilon = 0.75$ and reaches a maximum value $\varepsilon_m \cong 0.9$ (see Freeman, 1975 and references

[2] Note that due to a non-uniform rotation in the final state $E_{\mathrm{rot}\,eq}/E_{\mathrm{rot}\,i} \neq a_{eq}^2/a_i^2$.

therein). It is remarkable that ε_m is near to the theoretical upper limit $\varepsilon_{max} = 0.83$, which is given about in the framework of the model of a homogeneous spheroid with anisotropic pressure.

At the same time this model does not reproduce some other important properties of SO and disk galaxies. For instance, the rotational velocity (5) in the plane $z = 0$ increases monotonously and reaches its maximum value at the boundary surface. Such a behaviour of $\bar{v}_\varphi$ differs qualitatively from the typical rotational curve of galaxies which after its maximum is flat or declines.

This discrepancy between the model and the properties of real galaxies is due to the simplified assumption of homogeneity of the spheroid. Unfortunately, investigation of non-homogeneous models is rather difficult. However, it is possible to reach some qualitative conclusions. Obviously, concentration of mass to the centre of the spheroid makes the change of $\bar{v}_\varphi$ with the distance from the rotational axis nonmonotonous. At the same time, the account for non-homogeneous density distribution does not diminish, as compared with Eq. (15), the ratio E_{rot}/E_{rand} for the models where the distribution function $f(r, v)$ depends on two integrals of motion (J_z, E) and the density is constant on spheroidal surfaces. Therefore, a non-homogeneous spheroid with anisotropic pressure may be an attractive model for SO and S-galaxies and deserves further investigation.

As was mentioned above, the homogeneous model at any ε (including $\varepsilon \lesssim 0.7$) has too large a ratio E_{rot}/E_{rand} to be applied to elliptical galaxies. It is possible to diminish the ratio E_{rot}/E_{rand} only by means of global anisotropy in velocity dispersion such that

$$\frac{\overline{\langle v_r^2\rangle}}{\langle v_z^2\rangle} > 1. \tag{16}$$

However, it can be shown that, in the model considered, such an inequality is fulfilled only at $\varepsilon \gtrsim 0.63$. Hence, elliptical galaxies of all types ($E0$–$E7$) cannot be described by means of the homogeneous model investigated in the present paper. This is not surprising since the anisotropy given by Eqs. (3) is a special (and not the most general) case of anisotropy and is related to the assumption of homogeneity. Therefore, it would be premature to entirely reject the oblate spheroids as possible models for elliptical galaxies, because a spheroid with other kind of anisotropic pressure [such as the one given by Eq. (16)] may serve as a basis for an appropriate model. Indeed, there is observational evidence that some elliptical galaxies possibly are oblate spheroids (Kondrat'ev and Ozernoy, 1979). It is tempting to speculate that a non-homogeneous model of the spheroid with anisotropic pressure can provide the anisotropy in velocity dispersion possessing such a property as (16). Apparently, the distribution function $f(r, v)$ for a non-homogeneous spheroid with anisotropy (16) will depend, besides J_z and E, also on the third integral of motion, which is unknown, as distinct from the case of homogeneous spheroid [see Eq. (2)]. If it is so, then the non-homogeneous spheroid with anisotropy of the velocity dispersion might serve as an attractive model for both SO and E-galaxies.

5. Conclusions

The main result of this paper is that the maximum ellipticity for a homogeneous spheroid, with anisotropic velocity dispersion formed as a result of dissipationless collapse of a rotating sphere, is equal to $\varepsilon_{max} = 0.8321$. This shows that the value $\varepsilon_{max} = 0.7091$ obtained by Thuan and Gott (1975), for a homogeneous spheroid with isotropic pressure, reflects the specific properties of the model adopted not to mention that the Maclaurin spheroid cannot serve as a model for elliptical galaxies owing to their small rotation.

The value $\varepsilon_{max} = 0.83$ obtained above resembles the maximum ellipticity for lenticular and spiral galaxies. Possibly, other main properties of these galaxies may be explained, using the spheroids with anisotropic pressure, if we consider non-homogeneous models. At the same time, it is scarcely possible to consider the formation of *both* spheroidal and disk components of SO and S-galaxies as a result of dissipationless collapse. There is a number of well-known arguments which show that the formation of the disk component was accompanied by some dissipation of energy. If the formation of systems of galaxies (groups, clusters, etc.) took place by clustering of galaxies, the preservation of their individualities can give useful restrictions to the amount of energy dissipated (White, 1978).

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Appendix: Components of Velocity Dispersion in a Homogeneous Spheroid with Anisotropic "Pressure"

It is readily shown that the trajectory of a star moving in a plane which is perpendicular to z-axis is an ellipse:

$$2T_\perp r^2 - J_z - A^2r^4 = v_r^2 r^2, \tag{A.1}$$

where

$$T_\perp = (v_r^2 + v_\varphi^2)/2 + A^2r^2/2$$

and

$$J_z = r \cdot v_\varphi$$

are the first integrals of motion. Introducing the parameters of the ellipse $m = r_{max}^2$ and $n = r_{min}^2$ we obtain from (A.1)

$$v_r^2 = \frac{A^2}{r^2}(m - r^2)(r^2 - n). \tag{A.2a}$$

Further, from both $T_\perp$ and J_z one obtains the azimuthal component of the velocity:

$$v_\varphi^2 = m \cdot n \cdot \frac{A^2}{r^2}. \tag{A.2b}$$

The z-component of the energy integral, $T_z = v_z^2/2 + C^2z^2$, yields

$$v_z^2 = 2T_z - C^2z^2. \tag{A.2c}$$

By definition

$$\overline{v_r^2} = \frac{1}{\rho_0}\int\int\int v_r^2 f(r, v) d^3v. \tag{A.3}$$

To evaluate this integral, it is convenient to introduce new variables (m, n, T_z) instead of (v_x, v_y, v_z). This transformation has the Jacobian

$$J = \frac{A}{4}\frac{m - n}{(m \cdot n)^{1/2}}\{[2T_z - C^2z^2]^{1/2} \cdot [(m - r^2) \cdot (r^2 - n)]^{1/2}\}^{-1}$$

and the integral in (A.3) must be taken over the phase volume:

$$0 \leqq n \leqq r^2, \qquad m \equiv a^2(1 + 2T_z/C^2c^2),$$

$$\frac{1}{2}C^2z^2 \leqq T_z \leqq \frac{1}{2}C^2c^2(1 - r^2/a^2). \tag{A.4}$$

Note that the second of formulae (A.4) is nothing but the condition of touching the spheroid's boundary surface by all of stars, and this leads to a degeneracy of the distribution function $f(\boldsymbol{r}, \boldsymbol{v})$ [See Eq. (2) in the text].

Integrating in (A.3) we obtain the first of Eqs. (3) to be found. In the same manner, averaging (A.2c) in (3) over the phase volume, we obtain $\overline{v_z^2}$. To find σ_φ^2, we evaluate first $\overline{v_\varphi^2} = \sigma_\varphi^2 + \bar{v}_\varphi^2$:

$$\overline{v_\varphi^2} = \frac{1}{\rho_0}\int\int\int v_\varphi^2 f(\boldsymbol{r}, \boldsymbol{v})d^3v$$

$$= \frac{2A^2a}{\pi^2Cc}\cdot\int_{(1/2)C^2z^2}^{(1/2)C^2c^2(1-r^2/a^2)} \frac{dT_z}{(2T_z - C^2z^2)^{1/2}}$$

$$\int\int_0^{r^2} \frac{mn^{1/2}\delta[m - a^2(1 - 2T_z/C^2c^2)]}{[(m - r^2)(r^2 - n)]^{1/2}}\,dmdn$$

and then obtain Eq. (4). Similarly, one gets $\bar{v}_\varphi^2$.

To evaluate the components of the energy of chaotic motions, the components of the velocity dispersion obtained above must be averaged over the spheroid's volume. For example,

$$\langle\overline{v_r^2}\rangle = \frac{1}{V}\int\int\int_V \overline{v_r^2}dxdydz = 0{,}10A^2a^2. \tag{A.5a}$$

The averaging of the other components yields:

$$\langle\overline{v_z^2}\rangle = 0.20C^2c^2, \tag{A.5b}$$

$$\langle\sigma_\varphi^2\rangle = 0.06\,A^2a^2. \tag{A.5c}$$

Finally,

$$E_{\rm rand}/|W| = 0.50 - 0.30\left[\frac{1}{e^2} - \frac{(1 - e^2)^{1/2}}{e\cdot\arcsin e}\right] \tag{A.6}$$

which is shown in Fig. 1.

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